EXPLORATION GUIDANCE

Standard

MYP Mathematics

A concept-based approach

ENHANCED ONLINE

Years 4&5

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UNIT 1: Being specific

1.1 Problem-solving

Reflect and discuss 1

The students think about the following aspects about problem-solving:

- What makes a problem easy or challenging?
- Different kinds of problems. Ideas should include algebra problems (e.g. solving equations), geometry problems, word problems, etc.
- Different strategies that could be used to solve each of these types of problems (Can one single strategy solve any kind of problem? Why or why not?)

Students start a discussion about the different aspects of problem-solving.

Exploration 1 /Reflect and discuss 2

Students solve problems that require different methods and approaches, and link the steps in their solutions to Pólya's problem-solving steps. They should realize that they are already problem-solvers – everyone knows how to solve problems at some level.

Students then think about the importance of each of Pólya's problem-solving steps. Students typically focus almost exclusively on step 3 (Carry out your plan) and underestimate the importance of the other steps.

- If step 1 (Understanding the problem) is overlooked, students may not answer the question (despite some correct calculations being presented), or they may not finish their work by responding to the initial question. A question could also be misunderstood or misread, causing the solution to be off-topic.
- If step 2 (Devise a plan) is rushed, students may also forget to answer the question. Or they may lose track of their plan and go in a wrong direction when solving a problem. Spending time on this step helps students see the bigger picture and can help them come up with ideas for how they could solve the problem. It also helps them get a first estimate of what the answer could be, so that they could check that their final answer makes sense.
- If step 3 (Carry out the plan) is not done properly and mistakes are made, answers will be incorrect. Calculation errors are often an issue in this phase, as well as not fully understanding the methods used.
- If step 4 (Look back) is skipped, answers aren't checked and methods aren't validated. Students should compare their final answer with their ballpark estimate of their answer that they came up with in step 2 (Devise a plan). Intuition is also a helpful tool to raise alarms when mistakes are made – students should be encouraged to listen to their intuition when it tells them that something isn't right.

Reflect and discuss 3 and 4

Students take a minute to think about and compare the different examples and problem-solving strategies presented. They should think about the type of situation for which each strategy is the 'best' strategy.

Exploration 2/Reflect and discuss 5

Students are given a situation in which they are asked to both approximate and calculate more exact answers for the same problem. When giving estimates, students should keep the context of the real situation in mind. In the case of buying enough paint to paint a room, it is better to overestimate and have a bit of extra paint than to underestimate and run out of paint. However, it may be costly to greatly overestimate the amount of paint needed. Students should be mindful that the way they approximate should depend on the situation, and that some approximations are more reasonable than others.

UNIT 1: Being specific

1.2 The number system

Exploration 1

Students discover that the formula for the number of subsets of a given set *A* is 2^n, where *n* is the number of elements in *A*.

Reflect and discuss 1

Through the first bullet point and Exploration 1, students should understand why the empty set is a subset of every set.

Exploration 2

Venn diagrams are used to illustrate the subsets of the set of Real numbers and the IB notation for these subsets is shown. The definition of the complement of a set is helpful here.

Reflect and discuss 2

Students reflect on their findings from Exploration 2 by being prompted to think about the usefulness of the real number system, as well as the existence of other numbers that are not in this system. This provides an opportunity for the teacher to introduce other possible number systems, such as the set of complex numbers. Students may be encouraged to research number systems outside the real number system.

Exploration 3

Students explore other ways of classifying the subsets of the set of real numbers, e.g. tree diagrams and number lines. Then they create their own number line as an aid to order real numbers.

Reflect and discuss 3

Students explore the reasons for using the real number symbols for the different sets. They are once again encouraged to consider the existence of numbers outside the real number system.

Exploration 4

Students explore how to round numbers, first by using a number line, then without. They learn how to round numbers to different degrees of accuracy.

Significant figures are then introduced, starting with the most significant figure. Students truncate numbers to a given number of significant figures and determine the number of significant figures in different numbers.

Reflect and discuss 4

Students are made aware of the fact that when using rounded numbers to calculate, rounding errors are perpetuated and may cause the final answer to be quite far off from the exact answer. Students should think about when and how they round when using estimates to solve problems.

Reflect and discuss 5

Following on from Reflect and discuss 4, students should be made aware that sometimes estimates are necessary (e.g. money must be rounded to the nearest cent), and sometimes they are inevitable (e.g. lengths, distances and weights are always rounded to a certain degree of accuracy depending on the situation).

Reflect and discuss 6

Students reflect about different purposes that numbers serve. Whole numbers can be used for counting, labeling or ranking, but also for measurements. Irrational numbers, however, are generally only used when solving problems that involve measurement. Yet, people don't actually measure in terms of irrational numbers (e.g. you wouldn't measure a length to be $\sqrt{2}$ cm but you may measure it to be 1.4 cm. Rational and irrational numbers are thus often approximated to a decimal number that makes sense in a given situation.

Exploration 5

Students learn to estimate the value of a square root by placing it somewhere in between two consecutive whole numbers, depending on how many non-integer square root numbers are in between these two whole numbers. Students then compare their estimated values with calculator values, to realize that their estimate is only a rough estimate.

Attention is drawn to the fact that calculators also estimate square roots, depending on the number of digits that are displayed on their screen (or on the calculator settings for the number of digits to display).

Exploration 6

Students learn to write recurring decimals as fractions.

Attention is again drawn to the fact that calculators also round recurring decimal numbers, even though the exact numbers have an infinite number of decimal places.

Reflect and discuss 7

Students reflect on the distinction between an exact value (which can only be written as $\sqrt{3}$, for example) and an estimated value (which can be measured, such as 1.732).

Some rational numbers can also be written exactly (e.g. $\frac{2}{3}$) or estimated (e.g. 0.666667), while others can be written exactly both in fraction and in decimal notation (e.g. $\frac{1}{4} = 0.25$).

While exact answers are sometimes not meaningful (e.g. we understand 1.732 km but it is more difficult to mentally represent $\sqrt{3}$ km), they are more exact than estimated answers. Students should understand this and think about how they represent numbers when solving problems, and how to represent numbers in a solution.

UNIT 1: Being specific

1.3 Laws of exponents and scientific notation

Exploration 1

Students use their knowledge of exponents to study the behavior of exponents when the base and/or the exponent are positive or negative integers.

Answers:

1a 1 **b** −1 **c** $(-9)^7 = -9^7$ **d** $(-3)^{12} = 3^{12}$ **e** $5^2 \cdot 9^4$

2 When the exponent is even, the number is always positive (even with a negative base). When the exponent is odd, the number is the same sign as the base.

3a 2^5 **b**$(-7)^3 = -7^3$ **c** 3^3 **d** $14^0 = 1$

4 The product rule of exponents states that when multiplying exponents of the same base, the exponents are added. However, when multiplying by a number with exponent 0, the original number stays the same. Knowing that the multiplicative identity of any number a is 1, then if multiplying by a^0 is equivalent to multiplying by 1, then $a^0 = 1, a \neq 0$ must always be true.

5a 2^{-2} **b** 3^{-9} **c** 20^{-2} **d** $(-5)^{-3} = -5^{-3}$

6 With a similar reasoning as in question 4, if two exponents with the same base are divided, and the divisor has a larger power than the dividend, then the result is a fraction with a denominator that is a power of the base, and the exponents subtract to make a negative number. Thus $a^{-n} = \frac{1}{a^{-n}}, a \neq 0$.
A common misconception here is that a negative exponent gives a negative result. A reminder of the outcomes of questions 1 and 2 are appropriate here.

7a 0 **b** 0 c. no solution (it is impossible to divide by 0)

8 $0^a = 0$ for any value of $a > 0$. If $a < 0$ then 0^a doesn't exist, as we would be dividing by 0.

9a 1 **b** 1 **c** 1

10 $b^0 = 1$ for any value of b, $b \neq 0$. This was already discovered in questions **3** and **4**.

11 If $0^a = 0$ and $b^0 = 1$ for all $a, b \neq 0$, then it is impossible to know whether 0^0 would have a value of 0 or 1. Hence, it is indeterminate.

12 For $a^{-n} = \frac{1}{a^{-n}}$, if $a = 0$ then the result of a^{-n} would be a fraction where the denominator is a power of 0 (which would be equal to 0). Since it is impossible to divide by 0, $a^{-n} = \frac{1}{a^{-n}}$ must be true for all a as long as $a \neq 0$.

Reflect and discuss 1

Students determine when a negative base or exponent gives a negative result:

- If $a < 0$, then $a^b < 0, b \in Z$: sometimes true. It is true if b is odd, but not if b is even.
- If $a > 0$, then $a^b < 0, b \in Z$: never true. A number with a positive base can never be negative, even if the exponent is negative.
- If $b < 0, b \in Z$ then $a^b < 0$: sometimes true. It is only true if $a < 0$ and if b is odd. If $a > 0$ or b is even, then $a^b > 0$.
- If $b > 0, b \in Z$ then $a^b < 0$: sometimes true. It is only true if $a < 0$ and if b is odd. If $a > 0$ or b is even, then $a^b > 0$. (same reasoning as for the previous statement)

Conclusion: $a^b < 0$ only if $a < 0$ and if b is odd (at the same time). Otherwise (i.e. if $a > 0$ or b is even), $a^b > 0$.

Exploration 2

1

a Not in scientific notation. *a* = 13 140 000
b Is in scientific notation. *b* = 0.0000999
c Is in scientific notation. *c* = 7.24
d Not in scientific notation. *d* = 0.6205
e Is in scientific notation. *e* = 0.793
f Is in scientific notation. *f* = 20100
g Not in scientific notation. *g* = 370

2

a $b \times e = 7.92207 \times 10^{-5}$ easier in scientific notation
b $b + e = 0.7930999$ easier in standard form
c $a \div f = 6.5373 \times 10^2$ easier in scientific notation
d $a - f = 13119900$ easier in standard form
e $9d \times 8e = 3.5428 \times 10^1$ easier in scientific notation
f $9d - 8e = -0.7595$ easier in standard form
g $3c \times 2g = 1.60728 \times 10^4$ easier in scientific notation
h $3c + 2g = 761.72$ easier in standard form
i $a \div 100b = 1.3153 \times 10^9$ easier in scientific notation

j $a + 100b = 13140000$ easier in standard form

k $(a \times 150g)^2 = 5.318 \times 10^{23}$ easier in scientific notation

l $(a + 150g)^2 = 1.741 \times 10^{14}$ use a combination of standard form (for the addition) and scientific notation (for the squaring)

Reflect and discuss 2

Standard form is more useful to compare numbers that are comparable (within a certain range of powers of 10 of each other) and is more useful to add and subtract numbers.

Scientific notation is more useful to compare numbers that are far away from each other (difference of some powers of 10) and is more useful to multiply and divide numbers.

Scientific notation is used to manipulate extremely large or extremely small numbers, as is commonly the case in certain areas of science: at a cellular or molecular level, we are talking about extremely small numbers, whereas at astronomical levels, or when studying the number of cells in organisms or the number of molecules in a recipient, extremely large numbers are used. Hence the term 'scientific notation'.

Exploration 3

Millions and billions both feel like 'a lot', but they are not on the same scale. Although it is easy to state that 1 billion is 1000 times larger than 1 million, we get a better idea of the scale of each number when we compare them this way: 1 million seconds is equivalent to 11.57 days, whereas 1 billion seconds is equivalent to 31.7 years.

Reflect and discuss 3

While scientific notation helps us to manipulate very large or very small numbers, it is difficult to represent the actual size of such numbers simply by only looking at their scientific notation. Using other comparisons, such as time or distance or weight, can help us better understand the magnitude of very large or very small numbers.

One extension exercise would be to compare 1 million and 1 billion in as many ways as possible.

Another extension exercise would be to get students to ask (and then answer) questions where numbers of different magnitudes are compared, for example, 'How many 1€ coins would need to be piled up to reach the moon from Earth?'.

UNIT 1: Being specific

1.4 Units of measurement

Exploration 1

Students are to group units of measurement they come up with in different categories. Some obvious categorisations are

- by types of measurement (length, mass, time)
- by metric/non-metric measurements (meters, feet, liters, pints)
- by dimension (length, area, volume, capacity).

Reflect and discuss 1

There are often historical and cultural contexts involved in measurement. In the past, body parts, of the king for example, were used to measure lengths. With time and globalization, units of measurement have been standardized and a metric system (using powers of 10) has become more and more commonly used worldwide. But old habits die slowly, and many cultures are still attached to measurement systems other than the metric system.

In this reflection, students are asked to think about where units of measurement come from, and should understand that there is a conventional agreement on what these units are and represent.

Reflect and discuss 2

Following on from the previous reflections, students think about the importance of standardization when dealing with units that are used at a national or global level. Before measurements were standardized, units could vary, which causes all sorts of problems that students are asked to reflect on.

Exploration 2

The squares have the same area because their side lengths are equal. They are just measured in different units (1 m or 100 cm). Their area is thus also equal: 1 m^2 is equal to 10000 cm^2. This should help students visualize and understand equivalent areas in different metric units of measurement.

In the second part, the same process is repeated to find equivalent volumes in different metric units of measurement.

$$1\text{m}^2 = 10\ 000\text{cm}^2$$
$$1cm^2 = 100mm^2$$
$$1\text{m}^3 = 1\ 000\ 000\text{cm}^3$$
$$1cm^3 = 1000mm^3$$

Exploration 3

Students do the same calculation using different units of measurement:

1a. $30cm \times 50cm \times 100cm = 150000cm^3$
1b. $0.3m \times 0.5m \times 1m = 0.15m^3$
1c. The volumes are equal: $0.15m^3 = 150000cm^3$
2. The surface areas are also equal: $1.9m^2 = 19000cm^2$
3. The volume of the second cuboid is $2.50m \times 4.72m \times 5.45m = 64.31m^3$

Reflect and discuss 3

If different units of measurement can be used to calculate equal areas or volumes, then students should think about their choice of units. When using smaller units, they get more exact answers, but if the lengths are large, this may not make sense. When using larger units of measurement, lengths are usually approximated to some extent, which may cause approximation errors when calculating small areas of volumes. In the context of a problem, there is often a 'best' unit of measurement that corresponds to the size of the length, area or volume being calculated.

Reflect and discuss 4

Students compare converting currencies and converting units of measurement. They are similar because both are conversions. However, when dealing with currencies, exchange rates vary over time, so different rates can be used between two currencies – this isn't the case when converting units of measurement. Another difference is that when converting currencies, banks often charge an interest rate in addition to the exchange rate, in order to make a profit on currency exchange. This adds more volatility in the process of converting currencies. Unit conversions are fixed and don't change – they are not volatile conversions.

Exploration 4

Students research about natural units. Natural units are units of measurement based on universally physical constants. Examples of natural units are c (the speed of light), e (the unit of electric charge), and G (the gravitation constant). There are different systems of natural units (Planck units, Atomic units, etc).

The International Prototype of the Kilogram (IPK) is an object used to define the size of the kilogram since 1889. In 2019, the IPK was replaced by a new definition of the kilogram based on physical constants.

Natural units don't need a prototype as their 'prototype' already exists in nature.

Reflect and discuss 5

Students think about the difference between human-made and natural units of measurement. While one is an agreed upon by convention, the other exists in nature. Natural units are thus indisputable, whereas human-made units can (and often do) change over time.

Students then reflect upon the implications of this distinction between natural and human-made systems of measurement.

UNIT 1: Being specific

1.5 Surds, roots and radicals

Exploration 1

Students should find that a rational number is a number that can be written as a whole number or as a ratio between two whole numbers (i.e. a fraction).

Irrational numbers are numbers that cannot be written as a fraction, and when written in decimal form, the decimal part does not terminate nor is it periodic.

Reflect and discuss 1

Students should understand that it is not possible to take the square root of a negative number, as no number (positive or negative) will return a negative number when it is squared. Further discussion could be had about the imaginary number $i = \sqrt{-1}$ and complex numbers, which is a whole branch of mathematics that is 'imaginary,' that has its own set of rules and that has real-life applications.

Exploration 2

Students review how to estimate the value of an irrational number (i.e. by placing it in between two consecutive whole numbers).

It may be helpful to remind students about when to use exact numbers and when to use estimates.

Exploration 3

Students practise comparing numbers when they are written in different forms (whole numbers, fractions, square roots). Although some forms are more intuitive than others, it is good to develop the habit of comparing them to be able to determine whether a solution to a problem is reasonable or not.

Answers:

4

- **a** $\sqrt{5} < 5$
- **b** $\sqrt{5} < \frac{5}{2}$
- **c** $6 = \frac{18}{3}$
- **d** $\sqrt{6} < \frac{17}{3}$
- **e** $\sqrt{2} > \frac{7}{5}$

f $\sqrt{8} > \frac{141}{50}$

g $\sqrt{3} > \frac{433}{250}$

h $\sqrt{\frac{1}{2}} > \frac{1}{2}$

i $\sqrt{\frac{1}{2}} > \frac{7}{10}$

Reflect and discuss 2

Advantages of using radical notation are that the amounts used are exact. Disadvantages of using radical notation are that we don't easily have a sense (a mental image) of how big a number really is. For example, it is easier to have an idea of how far 8.5 km is than $\sqrt{72}$ km.

Radical numbers follow their own set of rules (rules of radicals) and when these rules are used, answers are exact. It is often possible to avoid using a calculator when working with numbers in radical notation, whereas working with decimal numbers can be more complicated to do without a calculator.

Exploration 4

Students discover the rules of radicals.

Answers:

1

- **a** 16
- **b** 49
- **c** 12
- **d** 36
- **e** 20
- **f** 1001

2 $(\sqrt{a})^2 = a, a \geq 0$

3

- **a** 6
- **b** 6
- **c** 5.477226
- **d** 5.477226
- **e** 28.913665
- **f** 28.913665

4 $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}, a, b \geq 0$

5

- **a** 0.666667
- **b** 0.666667
- **c** 0.39223227
- **d** 0.39223227

- **e** 1.7638342
- **f** 1.7638342

6 $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, a \geq 0, b > 0$

Exploration 5

Students use rules of radicals and the Pythagorean theorem to learn to simplify radicals, by visualizing the equivalence of $3\sqrt{2}$ and $\sqrt{18}$.

Reflect and discuss 3

Attention is drawn to the visual justification of the result in the previous exploration.

Exploration 6

Students compare radicals in unsimplified and in simplified form. They find how to determine when a radical is simplified or not and think about how to simplify it if it isn't (this is further shown in the following examples).

Reflect and discuss 4

Students compare the two methods shown for simplifying a radical number: different rules of radicals are used, but the result is the same.

Students then think about how simplifying radicals (finding square factors in a square root) compares to simplifying fractions (finding common factors between the numerator and the denominator).

Exploration 7

Students explore how to add and subtract radical numbers.

Answers:

1

- **a** Seven
- **b** Five
- **c** Eleven
- **d** Eight

2

- **a** $3 \cdot 2 + 4 \cdot 5 = 6 + 20 = 26$
- **b** $\frac{1}{3} + \frac{3}{4} = \frac{4}{12} + \frac{9}{12} = \frac{13}{12}$
- **c** $6x + 5y$

d $2\sqrt{2} + 2\sqrt{3}$

Draw attention to the fact that just like how fractions with different denominators or different variables simply can't be added or subtracted immediately, neither can different square roots.

3 Square roots can be added only when the radicand is the same in each term.

4 We are each time adding 3 objects to 8 more objects. These objects are *x*'s. 5's, tenths, and $\sqrt{2}$'s respectively.

5

a $\sqrt{5} + \sqrt{5} = 2\sqrt{5} = \sqrt{20}$

b All 3 expressions are equal to 4.472136 on the calculator

c The simplest form is $2\sqrt{5}$

6 $a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$ or equivalent

Reflect and discuss 5

Adding and subtracting radicals requires the radicands to be simplified. Radicals with equal radicands can be added or subtracted. Radicals with different radicands behave like unlike terms in a polynomial or like fractions with different denominators – they can't be added and must be expressed as the sum or difference between two terms.

Reflect and discuss 6

Students reflect about what it would mean to divide by an irrational number. Dividing evenly between a whole number of parts makes sense. How can we divide evenly between an irrational number of parts? This is a rationale for rationalizing the denominator – to make the divisor a whole number.

Exploration 8

1 Sums of irrational numbers are irrational. Products of irrational numbers can be rational or irrational.

2 Studying the product of rational and irrational numbers:

a Impossible - the product of two rational numbers can't make an irrational number.

b The product of two irrational numbers can be a rational number.
For example, $\sqrt{2} \times \sqrt{2} = 2$.

c The product of two irrational numbers can also be irrational (e.g. $\sqrt{2} \times \sqrt{3} = \sqrt{6}$).

3 Simplifying rationals allows us to work with smaller radicals, and may allow for adding and subtracting radicals when it isn't possible to add or subtract them in their unsimplified form (e.g. $\sqrt{12} + \sqrt{27} = 2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$). However, using unsimplified forms can be easier to multiply or divide without making mistakes (e.g. $\sqrt{20} \times \sqrt{30} = \sqrt{600} = 10\sqrt{6}$ may be easier to perform than $\sqrt{20} \times \sqrt{30} = 2\sqrt{5} \times \sqrt{30} = 2\sqrt{150} = 2 \cdot 5\sqrt{6} = 10\sqrt{6}$).

UNIT 1: Being specific

1.6 Absolute value

Reflect and discuss 1

When talking about differences in real-life situations, positive numbers are often used because the direction is often ignored or implied. For example, if two people discuss how much money is owed, the answer given will be a positive amount – it is assumed that both know who owes the amount to the other.

Another example is when comparing times in different time-zones. For example, Halifax is 5 hours behind Brussels. The word 'behind' implies that the time is 5 hours less there than the time in Brussels, but we do say that the time difference is 5 hours (and not negative 5 hours).

Exploration 1

1

a $x = 12$
b $x = 0$
c $x = 5$
d $x = \frac{3}{5}$
e $x = \frac{3}{5}$
f $x = 3.14159$
g $x = 3.14159$
h $x = 0.003$

All the answers are positive. The absolute value of a number is equal to that number if it is positive, or it is its opposite if the number is negative.

$$|x| = \begin{Bmatrix} -x, if\ x < 0 \\ x, if\ x \geq 0 \end{Bmatrix}$$

2

a 3
b 5
c 2
d 12
e 12
f $\frac{1}{2}$
g $\frac{1}{3}$
h a (as long as $a > 0$)

It is possible to find the absolute value of any number by taking the square root of its square.

$$|x| = \sqrt{x^2}$$

Reflect and discuss 2

- It isn't possible to take the square root of a negative number.
- Since the absolute value always returns a positive value, it is always possible to take the square root of the absolute value of a number.
- It is always possible to take the absolute value of any number.

Exploration 2

Students discover the properties of absolute values.

1

a Yes
b No
c No
d Yes
e No
f Yes
g Yes
h No

As the absolute value of a number is always positive, negative numbers can't be an absolute value of another number.

General property: $|a| \geq 0$

2

a $|a \times b| = |77| = 77$ $\quad |a| \times |b| = 11 \times 7 = 77$
b $|a \times b| = |-24| = 24$ $\quad |a| \times |b| = 6 \times 4 = 24$
c $|a \times b| = |-2| = 2$ $\quad |a| \times |b| = 8 \times \frac{1}{4} = 2$
d $|a \times b| = |3| = 3$ $\quad |a| \times |b| = 0.2 \times 15 = 3$
e $|a \times b| = |3\sqrt{3}| = 3\sqrt{3}$ $\quad |a| \times |b| = \sqrt{3} \times 3 = 3\sqrt{3}$
f $|a \times b| = |-1| = 1$ $\quad |a| \times |b| = \frac{1}{6} \times 6 = 1$

General property: $|a \times b| = |a| \times |b|$

3

a $\left|\frac{a}{b}\right| = \left|\frac{3}{10}\right| = \frac{3}{10}$ $\quad \frac{|a|}{|b|} = \frac{3}{10}$
b $\left|\frac{a}{b}\right| = \left|-\frac{2}{4}\right| = \frac{1}{2}$ $\quad \frac{|a|}{|b|} = \frac{2}{4} = \frac{1}{2}$
c $\left|\frac{a}{b}\right| = \left|-\frac{1}{5}\right| = \frac{1}{5}$ $\quad \frac{|a|}{|b|} = \frac{1}{5}$
d $\left|\frac{a}{b}\right| = \left|\frac{3}{30}\right| = \frac{3}{30} = \frac{1}{10}$ $\quad \frac{|a|}{|b|} = \frac{3}{30} = \frac{1}{10}$
e $\left|\frac{a}{b}\right| = \left|\frac{\sqrt{2}}{2}\right| = \frac{\sqrt{2}}{2}$ $\quad \frac{|a|}{|b|} = \frac{\sqrt{2}}{2}$
f $\left|\frac{a}{b}\right| = \left|\frac{0.2}{0.8}\right| = \frac{1}{4}$ $\quad \frac{|a|}{|b|} = \frac{0.2}{0.8} = \frac{1}{4}$

General property: $\left|\frac{a}{b}\right| = \left|\frac{a}{b}\right|$

4

a $|a^2| = |25| = 25$ $|a|^2 = 5^2 = 25$

b $|a^2| = |9| = 9$ $|a|^2 = 3^2 = 9$

c $|a^2| = \left|\frac{1}{4}\right| = \frac{1}{4}$ $|a|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

d $|a^2| = |6| = 6$ $|a|^2 = \sqrt{6}^2 = 6$

e $|a^2| = |11| = 11$ $|a|^2 = \sqrt{11^2} = 11$

General property: $|a|^2 = a^2$

5

a	$\|a^3\|$	$\|a\|^3$	a^3	$\|a^4\|$	$\|a\|^4$	a^4	$\|a^5\|$	$\|a\|^5$	a^5	$\|a^n\|$	$\|a\|^n$	a^n
2	8	8	8	16	16	16	32	32	32	2^n	2^n	2^n
−2	8	8	−8	16	16	16	32	32	−32	2^n	2^n	$(-2)^n$
−3	27	27	−27	81	81	81	243	243	−243	3^n	3^n	$(-3)^n$

General property: $|a^n| = |a|^n$

6

a	b	$\|a+b\|$	$\|a\|+\|b\|$	$\|a-b\|$	$\|b-a\|$	$\|a\|-\|b\|$
8	3	11	11	5	5	−5
6	−4	2	10	10	10	2
−1	9	8	10	10	10	−8
−5	−7	12	12	2	2	−2
$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$

7

Conclusions:

a $|a-b| = |b-a|$

b $|a+b| \leq |a| + |b|$

c $|a+b| \geq |a| - |b|$

d $|a-b| \geq |a| - |b|$

Exploration 3

Students discover how to find the absolute error, the relative error, and the percentage error of rounded numbers.

Reflect and discuss 3

- Different types of error are appropriate in different situations.
- The absolute error can be calculated by taking the larger value and subtracting the smaller value – the result will always be positive.
- Absolute error is not useful when it is necessary to know whether a rounded number is above or below a certain amount. Sometimes it is important to know if a number overestimates or underestimates an amount - in this case we wouldn't take the absolute value.

Reflect and discuss 4

Students reflect on the meaning of exact values, especially when dealing with measurements. While exact values can be calculated, they remain theoretical and can't be measured. Measurements will only be as accurate as the measuring tool used. Approximations are more common than exact values in real-life situations, but sometimes it may be interesting to find the error in rounding, for example to see how much material would be wasted or how much profit would be lost or gained.

UNIT 2: Decisions, decisions

2.1 Making generalizations

Reflect and discuss 1

There are lots of general statements that students might make – integer powers of 4 are always integers; the larger the exponent, the larger the value; and so on.

Their best observations might be that the final digits alternate between 4 and 6, which you should encourage them to refine to refer to even exponents and odd exponents. If they are happy to say that an even exponent always leads to an ending of 6, you might challenge them to consider 4^0, which does not fit the pattern. They can then consider limitations to their generalization.

Reflect and discuss 2

The specific problem is to simplify a numerical expression, $2014^2 - 2012^2$. The general expression finds the difference between any pair of squares with a difference of 2. It is easier in the sense that on this occasion, the algebraic expression is easier to manipulate squaring large numbers might be. It is also beneficial because a whole family of problems has been solved.

Exploration 1

In steps 1–3 students should find that they always obtain an even number. The diagram should lead them to observe that the configuration of dots always has a line of symmetry (the removed diagonal) and hence will always be even (since there is a one-to-one correspondence between dots above and below the line, regardless of the length of the edge of the square). In step 6, the expression $n^2 - n$ should be factorized to yield $n(n-1)$, which, being the product of two consecutive integers, will always be even.

Reflect and discuss 3

It is hoped that students will see that problems can have many possible solutions and that generalization and proof can take many different forms. A diagram alone will rarely constitute a complete proof, although with the right annotation, a diagram sometimes provides a clearer justification than long paragraphs.

Reflect and discuss 4

Is it necessary to experiment with numbers before generalizing? Strictly speaking, no. But students should be aware that exploring specific examples is often an important step when trying to form conjectures. If generalization is a tool which enables mathematicians to prove results, considering examples may be viewed as a heuristic tool which helps us to decide what result to prove in the first place. Furthermore, in some cases, specific examples provide insight into how a more general proof may be structured.

Exploration 3/Reflect and discuss 5

1 When $n = 4$, there are 8 regions and when $n = 5$ there are 16 regions.

2 Students might sensibly predict 32 regions when $n = 6$.

3 The diagrams offer 30 and 31 regions – 31 is, in fact, the maximum. The diagram on the left has fewer than the maximum only because three lines meet at the centre of the circle: note that in the second diagram, there are no places (other than on the circumference of the circle) where more than two lines meet.

In parts 4–7, students are guided further towards the false result (i.e. that the number of regions is given by 2^{n-1}) and subsequently to discover that it does not hold. In fact, the number of regions is given by $\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$

The exploration should reinforce the idea that whilst testing cases is useful, it is not the same as forming a proof, and that students must distinguish between exemplification and justification.

Reflect and discuss 6

If students were not previously convinced by the value of proof, the notion that encryption depends on knowing, with certainty, that a number is prime should certainly be of value. The hope is that they will accept that no amount of exemplification is equivalent to proof, and that there are many occasions upon which proof might be needed.

2.2 Coordinate geometry

Exploration 1

Students should arrive at the Pythagorean theorem, or use it if they already have learned it.

Reflect and discuss 1

Students are encouraged to research different proofs of the Pythagorean theorem. They explore the validity of its converse.

Reflect and discuss 2

Students are encouraged to explore the validity of the converse of true statements.

Exploration 2

Students will come up with the formula for the distance between two points. They will also see that the relative positions of the points do not matter.

Reflect and discuss 3

The distance formula always works for any two points. When two points are on the same horizontal line, one need only subtract the x-coordinates. When two points are on the same vertical line, one need only subtract the y-coordinates.

Reflect and discuss 4

Students should readily see that once two equal sides have been found, one can conclude that a triangle is isosceles. If a triangle is equilateral, then it is also isosceles, but the converse is not true.

Exploration 3

Students will discover the midpoint formula.

Reflect and discuss 5

Students reflect on the process of generalizing the rule for finding the midpoint of two points. In order to be able to use any generalization, it is important to verify its validity for all cases, or state cases where it is not valid.

Exploration 4

Students will quantify the gradient of any line by investigating the steepness of lines having the same ‘run’ but different ‘rise’. They will develop the formula for finding the gradient of any straight line using the coordinates of two points on the line. They should notice that the gradient is the same between any two points on the line. A rising slope has positive gradient, and a descending slope has negative gradient. The gradient expressed as a percentage is explored in travel signs.

Reflect and discuss 6

Students reflect on why gradients are expressed as percentages on roads, and in researching this, see that there are differences among countries. Students should discover that the tangent of the angle between the horizontal and vertical distances between two points also gives the gradient of the line.

Exploration 5

Students discover that the gradients of parallel lines are equal, and the gradients of perpendicular lines are negative reciprocals of each other.

Reflect and discuss 7

Students should notice that switching the positions of the points in using the gradient formula does not change the gradient of the line.

Exploration 6

Students discover that a horizontal line has zero gradient, and a vertical line has no gradient.

Exploration 7

Students discover the standard form of the equation of straight line $y - mx + c$, where m is the gradient and c the y-intercept.

Reflect and discuss 8

Students use the equation of a straight line to determine if a point lies on the line, by substituting the values of the coordinates into the equation.

Exploration 8

Students discover the point-gradient form of a straight line, and then convert to standard form.

Reflect and discuss 9 & 10

Students reflect on the conditions necessary to use the different forms of a straight line.

UNIT 2: Decisions, decisions

2.3 Modelling: Linear equations and systems of linear equations

Exploration 1

Students focus on the addition and multiplication principles to solve the equations.

Reflect and discuss 1

Exploration 1 highlighted why the addition and multiplication principles are called equivalence transformations. Using these principles result in equivalent equations.

Reflect and discuss 2

Students realize that multiplication by 0 is not an equivalence transformation. They should reflect on other operations they know and determine if they are equivalence transformations, e.g., subtraction, taking the square root, squaring, etc. They should see that division is an equivalence transformation, except division by 0. Squaring is not an equivalence transformation, as extraneous solutions may occur.

Reflect and discuss 3

Using a graph to solve an equation is not always reliable, in particular when there are non-integer solutions.

Reflect and discuss 4

When solving a linear system of equations in two variables, any variable may be substituted. However, one variable might be easier to substitute than the other.

Exploration 2

Students are led to use the elimination method to solve a system of linear equations.

Exploration 3

Students solve the same system of equations by eliminating one variable first. Then they solve the same system by eliminating the other variable. They recognize that it is sometimes easier to eliminate one variable rather than the other, but the answers come out the same regardless.

Reflect and discuss 5

Students learn to recognize which variable would be easiest to eliminate in order to solve a system of equations using this method.

Reflect and discuss 6

Students reflect on which method of solving a system of linear equations is the most accessible in a given problem.

Exploration 4

Students will discover that a system of two linear equations:

- Has one unique solution, an ordered pair, if their gradients and y-intercepts are the same.
- Has no solution if their gradients are the same but their y-intercepts are different.
- Has an infinite number of solutions if their gradients and y-intercepts are the same.

When manipulating equations to solve the system, an arithmetical mistake may be made. Therefore solutions should always be tested in the original solutions.

Reflect and discuss 7

Students create a mathematical model for a real-life example, and use a system of equations to solve it. They are encouraged to reflect on the economics of saving money.

Reflect and discuss 8

Students should realize that it is possible to create more than one correct mathematical model for a real-life problem. Although the methods to solve them might be equivalent since they produce the same result, the methods are not equal, as some may be more efficient than others.

UNIT 3: Back to the beginning

3.1 Relations and functions

Exploration 1

Students explore sets, elements (or members) of a set, ordered pairs, and mapping diagrams by using concrete examples from the classroom. This is an introduction to number sets and to the vocabulary related to sets.

In this exploration, relations are given as a list of ordered pairs.

Exploration 2

Students explore sets and mapping diagrams for mathematical relations, representing them in the same way the sets were represented in the relations of Exploration 1. In addition, ordered pairs are represented as coordinates and plotted on a coordinate plane. In cases where a set has an infinite number of elements, it is impossible to list them all – yet the relation can still be represented on a coordinate plane.

In this exploration, relations are given as a specific rule.

Reflect and discuss 1

Students reflect on the similarities and differences of representing a relation using mapping diagrams or using a graph. Mapping diagrams can be useful to visualize the sets A and B, as long as there are not too many elements in finite sets, otherwise they become cumbersome. Graphs are a way of visualizing a relation, but it takes a closer look to determine what the elements of each set are.

Reflect and discuss 2

The definition of a function is given a few pages earlier. Before exploring functions further, students reflect on the following facts about functions and relations:

- Every function is a relation (a function (which is a relation) maps elements of one set (the domain) to elements of another set (the range)).
- Not every relation is a function (some relations are one-to-many or many-to-many, whereas a function is a type of relation that is either one-to-one or many-to-one).

Exploration 3

By relating graphs to mapping diagrams, students discover how the vertical line test on a graph can determine whether a relation is a function or not.

Reflect and discuss 3

- Different ways of representing relations and functions are: lists of ordered pairs, mapping diagrams, equations, graphs.
- Mapping diagrams can show which relations are functions (one-to-one and many-to-one) or not (one-to-many and many-to-many). Graphs of relations can do the same by use of a vertical line test (a relation is a function if no vertical line intersects with the graph of the relation more than once).
- In math class and when solving real-life problems, we often focus more on functions than on relations, even if functions are a subset of relations. With a function, there can only be one answer for each output, meaning that each x-value from the domain has at most one corresponding y-value in the range. This is particularly important in real life problems. For example, what would happen if for two investments of the same amount, same rate of return, and same time frame, there were two different returns? Or if you could be at two different places at the same time? Or if a child could have different biological mothers?

Reflect and discuss 4

- Justification is used whenever reasons are given to support an answer.
- Determining whether a relation is a function is a claim that can be supported by a justification.
- Justifying whether a relation is a function or not helps us understand what a function is and also helps us determine more easily which relations are functions and which ones aren't.
- Examples of relations that are not functions are $x = 2$ (or any vertical line) and $y^2 = x$, which when solved for y is $y = \pm\sqrt{x}$. Usually equations that are not functions involve an even power of y.

Exploration 4

1 This is a function because no matter what the initial number is, there is only one final answer.

2 $f(x) = (2x + 3)^2$

3 Various student answers. Students practice writing verbally constructed equations (this may be a good opportunity to remind students about order of operations).

Reflect and discuss 5

Students come to realize that when using equations to model real-life situations, there may be restrictions on the domain and/or range.

Exploration 5

Students discover what it means to evaluate a function (finding *y* when *x* is given) and solving an equation (finding *x* when *y* is given).

The linear function is a one-to-one type of function – the equation has one solution for every single value of $x \in \mathbb{R}$.

The quadratic function is a many-to-one type of function – the equation may have up to two solutions. In fact, a quadratic equation may have no solution, one or two solutions.

Exploration 6

Students explore different relations and functions to find that a function can be written as $f(x)=...$

Reflect and discuss 6

Students think about the following aspects about relations and functions:

- One-to-many and many-to-many relations generate more than one output value for a single input value.
- Relations that aren't functions have more than one output value for a single input value. They do not pass the vertical line test.

UNIT 3: Back to the beginning

3.2 Quadratic expressions

Exploration 1

The objective is for students to identify that in each quadratic expression, there exists a variable and that the square of that variable appears as the term of highest order (power). You might sensibly ask students whether writing the expression in a particular way (i.e. in descending powers) makes it easier to recognize quadratic expressions.

The term 'expression' can be hard to define in simple terms for students. 'Something which represents a value' is a good starting point, though more formally, it is a string of mathematical symbols which obeys correct syntactic rules. If you have developed a sense amongst your students that equations and inequalities are sentences (since they represent facts, and possess verbs – 'to equal', 'to be greater than') then expressions are somewhat akin to noun phrases.

Exploration 2

A key objective of this exploration is to enable the students to practice expanding brackets to form quadratic expressions. In addition, it is hoped that their attention will be focused on key results.

1

Factorized expression	p	q	Expanded expression	a	b	c
$(x+3)(x+5)$	3	5	$x^2+8x+15$	1	8	15
$(x+2)(x+9)$	2	9	$x^2+11x+18$	1	11	18
$(x-3)(x-6)$	−3	−6	$x^2-9x+18$	1	−9	18
$(x+3)(x-4)$	3	−4	x^2-x-12	1	−1	−12
$(x-3)(x+4)$	−3	4	x^2+x-12	1	1	−12
$(x+2)(x-5)$	2	−5	$x^2-3x-10$	1	−3	−10
$(x-8)(x+4)$	−8	4	$x^2-4x-32$	1	−4	−32
$(x-1)(x+6)$	−1	6	x^2+5x-6	1	5	−6

2 Students will hopefully notice that $b=p+q$ and $c=pq$. Furthermore, they might notice that this means that if c is negative then p and q will be of different signs. They should also observe that a is always 1 - though they might not mention this. It is worth reminding them of the non-monic quadratic expressions presented in exploration 1 if they miss this detail.

3 It is hoped that students will leap straight to writing:

Factorized expression	p	q	Expanded expression	a	b	c
$(x+1)(x+7)$				1	8	7
$(x-3)(x+8)$				1	5	−24
$(x-1)(x-2)$				1	−3	2
$(x+5)(x-5)$				1	0	−25

4 Remember that 'verify' is part of the level descriptors for Criterion B, Investigating Patterns. Students should be able to make predictions based on patterns as above, and then check their predictions by (in this case) expanding the original expressions.

5

Factorized expression	p	q	Expanded expression	a	b	c
$(x+2)(x+2)$	2	2	x^2+4x+4	1	4	4
$(x+4)(x+4)$	4	4	$x^2+8x+16$	1	8	16
$(x-1)(x-1)$	−1	−1	x^2-2x+1	1	−2	1
$(x-6)(x-6)$	−6	−6	$x^2-12x+36$	1	−12	36

6 There are several additional patterns that the students might comment on in the case of these perfect squares: c is always positive (and the square of p or q); b is always even (and twice p or q).

Exploration 3

This brief exploration introduces students to quadratic expressions which lead to the difference of two squares. They should note that where $p=-q$, the resultant quadratic contains no linear term.

Reflect and discuss 1

Students should observe that the first diagram is a large square of area a^2 but that an area equivalent to b^2 has been left unshaded. Since the pale blue rectangle has height $a-b$, which is the same as the width of the dark blue rectangle, rotating the latter creates a single rectangle of length $a+b$ and width $a-b$. Hence the original area, a^2-b^2 is equal to the area $(a+b)(a-b)$, as required.

Reflect and discuss 2

Students can verify quickly that Aishah's factorization is incorrect by expanding her brackets. The key detail that they should notice is that the quadratic in question does not fit the model of the quadratics in Exploration 2 because the coefficient of x is not equal to 1, i.e. the quadratic is non-monic.

Reflect and discuss 3

Example 5 is easier because 3 is prime and all the coefficients are positive; so there are far fewer permutations to consider. That said, students with quick mental arithmetic may yet find that they are able to factorize expressions such as that in Example 6 quickly and simply by inspection and there is no reason why they shouldn't do so. The majority of students will find methods such as those in Exploration 5 helpful.

Systematic listing has the potential to be time consuming whenever there are many possibilities to consider.

Exploration 4

The table for step 2 should read:

$(2x+3)(x-4)$	$2x(x-4)+3(x-4)$	$2x^2-8x+3x-12$	$2x^2-5x-12$
$(2x-3)(x+7)$	$2x(x+7)-3(x+7)$	$2x^2+14x-3x-21$	$2x^2+11x-21$
$(3x+2)(2x-1)$	$3x(2x-1)+2(2x-1)$	$6x^2-3x+4x-4$	$6x^2+x-4$
$(4x-2)(x+3)$	$4x(x+3)-2(x+3)$	$4x^2+12x-2x-6$	$4x^2+10x-6$
$(3x+4)(2x-1)$	$3x(2x-1)+4(2x-1)$	$6x^2-3x+8x-4$	$6x^2+5x-4$

…but it is very difficult for students to complete the bottom line unaided. You may wish to interrupt them before they complete it. Going to the left from the right-hand column is difficult because there are infinitely many different ways of splitting the linear term.

In question 5, Carmen obtains $3x(2x+7)+2(2x+7)$ as the next step, whereas Miranda obtains $2x(3x+10) + 5x+14$. Since Miranda's expression does not contain a common factor, Carmen's is more useful.

Exploration 5/Reflect and discuss 4

Students should add as many rows as they need and you should encourage them to vary the signs within the brackets. The coefficients of the split middle term should have sum b and product c.

Exploration 6/Reflect and discuss 5

Step 1 contains expressions of the forms $a^2 - b^2$ and $a^2 + b^2$. Students should observe that the latter cannot be factorized, though their explanations may well be limited. A good response would notice that in order to have no linear term, the signs in the brackets must be opposite – but that this on turn would lead to a negative constant term.

In Step 3, all five expressions can, in principle, be factorized, only not necessarily using integers, as the brackets in Step 5 suggest.

Students should be left with an appreciation that irrational numbers can sometimes be used to generate expressions with rational coefficients.

3.3 Representing quadratic functions

Reflect and discuss 1

Students are asked to think about the number of points needed to determine a unique quadratic. They probably will not come upon the fact that the answer is 3, but this will be revisited again later in this section.

Exploration 1

This exploration is best done using graphing technology, or a GDC with slider bars. Students will explore the effects of varying the parameters (coefficients) a, b, and c on the graph of a quadratic function.

- Changing the value of a affects the width of the graph.
  - If $a > 0$, the parabola is concave up
  - If $a < 0$, the parabola is concave down.
  - If $a = 0$, the parabola becomes a straight line. (Remind students that this is the reason that in the definition of a quadratic, a is not allowed to be 0.)
- Changing the value of b is not so clear, and students might need some help in what to look for. They should be encouraged to observe the shape formed when moving the b slider in relation to the original parabola, namely another parabola! Changing the value of b will move the axis of symmetry from side to side; increasing b will move the axis in the opposite direction. In other words, the parabola formed by moving the b slider can be described as following:
  - If $b > 0$, the new parabola moves to the left and down (if $a > 0$) of the original parabola.
  - If $b = 0$, the new parabola does not move.
  - If $b < 0$, the new parabola moves to the right and down (if $a > 0$).
- Changing the value of c moves the parabola up or down.
  - If $c > 0$, the parabola moves up.
  - If $c < 0$, the parabola moves down.

Reflect and discuss 2

The fact that the parameter a in a quadratic cannot be 0 is emphasized. They reflect on the mathematical meaning of dimension in coordinate geometry.

Exploration 2

Students arrive at the formula for the axis of symmetry of a parabola, and the coordinates of the vertex. They observe the relationship between factors of a quadratic and its *x*-intercepts.

Reflect and discuss 3

Students should conclude that the graph of a factorable quadratic has *x*-intercepts that are rational numbers. Quadratics could however have irrational *x*-intercepts (non-factorable quadratics), as well as no *x*-intercepts. In the case of the latter, they should conclude that the graph of the quadratic would either be above or below the *x*-axis, i.e., would not intersect the *x*-axis.

Reflect and discuss 4

Students should realize that since the *y*-value of a function at the *x*-intercept of its graph is 0, the *x*-intercepts are also referred to as the zeros of the function. Students should observe that in order to find the vertex when the quadratic is given in factored form, they would need to change the quadratic to standard form and then use the formula for finding the axis of symmetry. The coordinates of the vertex are therefore $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Exploration 3

Students should observe that the general form of a quadratic whose graph has only one unique *x*-intercept, (*h*, 0), is $y = a(x-h)^2$.

Exploration 4

Students should observe that for a quadratic to have one unique factor, $c = \left(\frac{b}{2}\right)^2$.

Reflect and discuss 5

Students consider the advantages in using the standard form or vertex form to graph a quadratic function. They should now realize that three points are necessary to determine a unique quadratic.

Reflect and discuss 6

This further emphasizes that three points are necessary to determine a unique quadratic.

Reflect and discuss 7

Students investigate the features of the graph of a quadratic in its different forms.

Reflect and discuss 8

Students should realize that a mathematical model is an approximation to a real life object or problem.

Exploration 5

Students find a mathematical model for the Golden Gate Bridge, and compare their models with others students. There should be similarities and differences in the models that they come up with, which makes the basis of a good discussion.

Exploration 6

Students should find objects to model mathematically using quadratic functions.

Reflect and discuss 9

Students should have found many objects that have a parabolic shape. This can lead to a discussion as to why objects have parabolic shapes, such as satellite dishes, sound reflectors, the Whispering Gallery at St Paul's Cathedral in the UK, solar cookers.

UNIT 3: Back to the beginning

3.4 Solving quadratic equations

Exploration 1/Reflect and discuss 1

Students graph the factors of a quadratic as separate linear functions, and the quadratic, on the same graph. They should observe that the *x*-intercepts of the linear functions and the quadratic function are the same.

Exploration 2

Students should discover that the solutions of a quadratic equation equal to 0 are the same as the solutions of the linear factors when set equal to 0.

Reflect and discuss 2

Using the given functions, students should realize that a quadratic cannot have more than two unique solutions.

Reflect and discuss 3

Students should realize that a quadratic equation can have no solutions. This would mean that its graph is entirely above or entirely below the *x*-axis.

Exploration 3

Using dynamic geometry software, students can quickly observe that

- A quadratic is always the product of two linear functions, except when the linear functions are constant functions, e.g., $y = 2$.
- If the quadratic is entirely above or entirely below the *x*-axis, it does not have zeros.
- The number of zeros of a quadratic whose vertex is on the *x*-axis is two equal zeros.

Reflect and discuss 4

Students observe that quadratic equations of the form $x^2 - k = 0$ can easily be solved by rearranging the equation in the form $x^2 = k \Rightarrow x = \pm\sqrt{k}$.

Reflect and discuss 5

Through the given example, students realize that “squaring” is not an equivalence transformation since the equation can have extraneous solutions. When using this method, therefore, they must check all solutions in the original equation and reject those that are extraneous.

Exploration 4

This exploration introduces students to the discriminant of a quadratic function. The discriminant is used to determine the number of real solutions of a quadratic equation. It also provides a check as to whether or not a quadratic is factorable.

Reflect and discuss 6

Students can discuss which of the three methods of solving quadratic equations they prefer, and why. They should also discuss whether or not all three methods are always applicable to every quadratic equation.

UNIT 4: Mathematically speaking

4.1 Set operations and Venn diagrams

Exploration 1

Students will form the sets resulting from the set operations of union and intersection, without having the symbols for these operations.

Reflect and discuss 1

Students will realize the convenience of having symbols for set operations.

Reflect and discuss 2

From the previous exercises, students will understand what a subset means and therefore will see that the intersection of two sets is a subset of their union.

Exploration 2

Through creating Venn diagrams, students will derive De Morgan's Laws on sets.

Reflect and discuss 3

Students' knowledge of De Morgan's Laws is re-emphasized.

Exploration 3

Students explore the validity of the properties of real number operations applied to set operations.

Exploration 4

Students discover the addition rule $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, and apply it to the given problem in step 2. At Standard Level, this rule is only required for mutually exclusive sets, that is, when the intersection of sets A and B is 0. Therefore, although the rule is not necessary to solve Venn diagram problems for sets that are not mutually exclusive, students might find it useful.

Reflect and discuss 4

This solidifies the usefulness of the addition rule.

Reflect and discuss 5

This affords an opportunity for group work to employ sets and their operations in a variety of real life situations.

UNIT 4: Mathematically speaking

4.2 Probability of single and combined events

Exploration 1

Students label the given events from impossible to certain in order to get a feeling for the numerical probability of an event happening.

Reflect and discuss 1

Students should come up with their own events whose probability is 0 (an impossible event) and another event whose probability is 1 (an event certain to happen).

Exploration 2

Students test a given probability of a real-life problem by simulation, either by hand or using their GDC. They will run the simulation a certain number of times so that they can see that the probabilities of the same event happening vary in a real-life situation. This highlights the meaning of theoretical versus experimental probability. They reflect on their results by answering the prompts.

Reflect and discuss 2

Students can simulate the rolling of the die on the GDCs rather than by hand. The number of winners among 8 people is $1\frac{1}{3}$, hence 1 winner.

Exploration 3

Although the MYP standard framework does not include the notion of independent events and the multiplication rule (for two independent events A and B, $P(A \cap B) = P(A) \times P(B)$), students are led to generate this result from the given problem. Using the table they create, they see that the P(1,T) = $\frac{1}{8}$.Then, P(1) = $\frac{1}{4}$ and P(T) = $\frac{1}{2}$. Multiplying P(1) with P(T) gives the same result as P(1,T), that is, the multiplication rule for two independent events. The rule may come in handy, but is not absolutely necessary, since the same result can be found without it.

Reflect and discuss 3

This refers to Example 3 preceding it. Students should see that for two independent events (the teacher should ensure students know the difference between independent and dependent events):

The order in which the branches of the trees are done is irrelevant.

The probabilities of the intersection of events (denoted by the use of the word ‘and’) is found by multiplying across the branches.

The probabilities of the union of events (denoted by the use of the word ‘or’) is found by adding the results on the last branches. When adding all the results on the last branches, the number 1 should be obtained, as this is the probability of the entire sample space.

Exploration 4

Students explore the probability of two events happening when the outcome of the second event is dependent upon the outcome of the first event. They should realize that the sample space is decreased by 1 for the second branch of the tree diagram, when there is no replacement.

Reflect and discuss 4

Students compare their tree diagrams or table of outcomes with other students, and discuss any variations that arise. They can reflect on which scenarios (either the tabular or tree diagram) is preferable in any given situation.

Exploration 5

This exploration sets the stage for calculating probabilities of mutually exclusive events. Students reflect on which kinds of events cannot happen at the same time, and should conclude that the probability of mutually exclusive events happening is the sum of their individual probabilities.

Reflect and discuss 5

Students reflect on the appropriateness of the term ‘mutually exclusive events’.

Exploration 6

Students should conclude that the more times an experiment is conducted, the experimental probability approaches the theoretical probability.

Reflect and discuss 6

Students attempt to design a game where everyone has an equal chance of winning (a fair game), and another game where this is not the case. The teacher might like to expand on the notation of a fair game.

UNIT 5: Spacious interiors

5.1 Surface area and volume

Exploration 1/Reflect and discuss 1

This exploration is to remind students that an *area* can be calculated for 2D figures, whereas a *surface area* is calculated for 3D figures (by adding the areas of all its faces).

Exploration 2

Students remember the properties of different 3D figures: prisms, cylinders, pyramids and cones.

- Prisms and pyramids both have bases that are (often regular) polygons.
- Cylinders and cones have bases that are circles.
- Prisms and cylinders have cross-sections that are congruent to their bases.
- Pyramids and cones have cross-sections that are similar to their bases.
- The sides of a prism are rectangles, the sides of a pyramid are triangles that meet at the apex.
- Cylinders and cones don't have sides. Instead, they each have one curved surface.
- A cylinder is like a prism with a circular base. A cone is like a pyramid with a circular base.

Reflect and discuss 1

A sphere is also a 3D solid but it does not have any faces. It has similar cross-sections (like a pyramid or a cone), and the cross-section that passes through the center of the sphere is comparable to its base.

Finding the volume and surface area of a sphere involves integral calculus. Archimedes used a method very similar to integral calculus to compare the volume of the sphere with the volume of a cylinder. It is known as his greatest achievement.

One way to find the volume of a sphere is to measure a volume of water in a receptacle, and to measure it again when the sphere has been completely immersed in the water. The difference in the two volumes is the volume of the sphere. This experiment is carried out in Exploration 4.

Exploration 3

Students explore different methods of calculating the surface area of a square-based pyramid based on the information that is given.

1 A square of side length 10 cm and four identical isosceles triangles with a base of 10cm and side lengths 7 cm.

Area of a square: side length ×side length

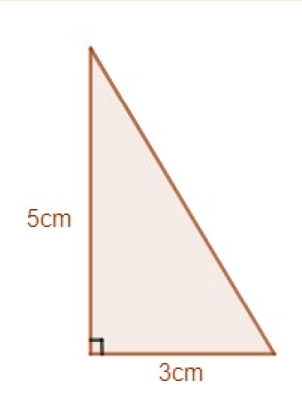
Area of an isosceles triangle: base length× height, where $\text{height} = \sqrt{(\text{side length})^2 - (\text{half the base})^2}$

2 Area of the base: $100\ \text{cm}^2$

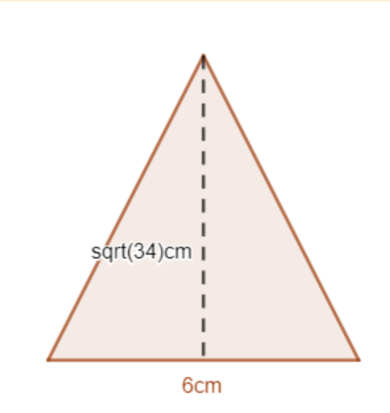
Area of one of the triangles: $5\text{cm} \times \sqrt{(7\ \text{cm})^2 - (5\ \text{cm})^2} = 5\sqrt{24}\ \text{cm}^2 = 10\sqrt{6}\ \text{cm}^2$

Surface area of Pyramid A: $100\ \text{cm}^2 + 4(10\sqrt{6}\ \text{cm}^2) = 197.98\ \text{cm}^2$

3 In the right triangle, the base is 3 cm (half of the length of the square base of the pyramid), and the height is 5 cm. The slant height is the hypotenuse of this triangle.



Using the Pythagorean theorem, the slant height of this triangle is $\sqrt{34}\ \text{cm}$.



The area of one of the triangular faces of the pyramid is $3\sqrt{34}\ \text{cm}^2$.

The total surface area of pyramid B is $6\ \text{cm} \times 6\ \text{cm} + 4 \times 3\sqrt{34}\ \text{cm}^2 = 105.97\ \text{cm}^2$

Reflect and discuss 2

In a square-based pyramid, all the triangular faces are congruent.

In a rectangular-based pyramid, the faces are not all congruent. Opposite faces are congruent to each other, but adjacent faces are not congruent.

In a pyramid where the base is any regular polygon, the triangular faces are all congruent.

Students think about how to find the area of a regular hexagon, and how to find the slant height of a pyramid with a base that is a regular hexagon. A regular hexagon can be divided into six equilateral triangles, which is a helpful tip. Drawings are recommended to help find surface area of a pyramid with a regular hexagon as its base.

Reflect and discuss 3

Since the surface area of a circle is $4\pi r^2$, and the area of a circle with the same radius is πr^2, the orange peels should fill exactly four complete circles.

The image of orange peels covering four circles of the same size as the orange develops a visual understanding of the surface area of a sphere.

Exploration 4

Students use a golf ball and a cylinder filled with water to measure the water that is displaced when a golf ball is put into the cylinder.

Part 1: By measuring the volume of water the golf ball displaces, students find the volume of a sphere.

Part 2: Students calculate the volume of a cylinder where the height and diameter are both equal to the diameter of the sphere. Students then compare this volume with the volume of the water displaced by the sphere. They should find that the sphere's volume is $\frac{2}{3}$ of the volume of the circumscribed cylinder.

UNIT 5: Spacious interiors

5.2 Geometric transformations

Exploration 1

Students manipulate physical shapes to come up with the different ways these shapes can be transformed without changing the shape. They can be displaced (translated), turned (rotated) and flipped (reflected).

They first think about the pieces of information that are required in order to perform the different transformations:

- A translation requires a new position (the number of units in the *x*-direction and the number of units in the *y*-direction).
- A rotation requires an angle of rotation, and perhaps a center of rotation.
- A reflection requires a mirror line.

They then think about what changes and what doesn't change for each of these transformations:

Transformation	**What changes**				
	position	**direction**	**orientation**	**shape**	**size**
Translation	Y	N	N	N	N
Rotation	Y or N	Y	N	N	N
Reflection	Y	Y	Y	N	N

Lastly, students think about a new type of transformation: enlargements.

- An enlargement requires a scale factor. (The center of enlargement is in the extended book)

Transformation	**What changes**				
	position	**direction**	**orientation**	**shape**	**size**
Translation	Y	N	N	N	N
Rotation	Y or N	Y	N	N	N
Reflection	Y	Y	Y	N	N
Enlargement	Y or N	N	N	N	Y

The major difference between an enlargement and the other transformations previously seen is that an enlargement changes the size of a shape. Hence, enlarged shapes are similar but not congruent to the original shape, whereas translated, rotated or reflected shapes are all congruent to the original shape.

Exploration 2

1 and **2**. Students compare different types of tessellations:

Figure A: the tiles are all identical regular polygons.
Figure B: the tiles are all regular polygons (but not all the same ones).
Figure C: the tiles are different shapes that are not regular polygons.

3 Focusing on tessellations made from regular polygons, students look at their interior angles to determine which regular polygons tessellate or not.

4 Students then learn how to label a vertex based on the tiles (regular polygons) that surround it, and compare the labelling sequences of two different tessellations.

5 Students play around with graphing technology to come up with regular and semi-regular tessellations (explained just after the exploration).

Reflect and discuss 1

Equilateral triangles tessellate because their interior angle (60°) fits exactly 6 times in 360°.
Squares tessellate because their interior angle (90°) fits exactly 4 times in 360°.
Regular hexagons tessellate because their interior angle (120°) fits exactly 3 times in 360°.
No other regular polygon has an interior angle that fits an exact number of times in 360°.

To know whether shapes tessellate, the sum of all the angles at each vertex must be 360°. Thus, knowing the angles of each shape helps us calculate the sum of the angles at each vertex. If they do not add up to 360°, they do not tessellate.

Exploration 3

Students discover that it is possible to obtain one single transformations by using other transformations.

1 A translation of $\begin{pmatrix}4\\3\end{pmatrix}$ can be obtained by performing

- 2 rotations
- 2 reflections (or an even number of reflections)
- A combination of an even number of reflections and one or more rotations

2 A reflection in the line $y = x$:

- cannot be obtained by performing only rotations
- can be obtained by an odd number of reflections
- can be obtained by a combination of an odd number of reflections and rotations.

3 A rotation of 90° clockwise around the point (1,1) can be obtained by performing:

- any number of rotations
- two reflections (or an even number of reflections)
- a combination of an even number of reflections and one or more rotations.

Exploration 4

- A translation can be obtained by performing two reflections. An example of these two reflections in this case: a reflection in the line $y = -3x + 16$ followed by a reflection in the line $y = -3x + 26$. Other solutions possible.
- A rotation can be obtained by performing two reflections. The easiest example is a reflection in the line $y = -2$ followed by a reflection in the line $x = 3$. Other solutions possible.
- A glide reflection can be obtained by performing three reflections. One example of these three reflections in this case is a reflection in the line $y = 2$, then a reflection in the line $y = 2x - 2$ followed by a reflection in the line $y = 2x - 7$. Other solutions possible.
- Conclusion: Any isometric transformation can be obtained by performing 1, 2 or 3 consecutive reflections:
  - one reflection for a reflection
  - two reflections for a translation or a rotation
  - three reflections for a glide reflection

Exploration 5

Students compare functions transformations with geometric transformations.
This table is an example of the type of comparisons that students can make.

	function	geometry
input	x	An original shape x
output	$y = f(x)$	An image of a shape y
a function that maps elements from one set to another	$f(x)$ maps x onto y	A transformation T maps the original shape x to its image y
Each element in the domain matches one and only one element in the domain	If f is a function, then each element x in the domain maps onto one and only one element y in the range (this is what makes a relation a function)	If T is a transformation, then each shape x in the domain maps onto one and only one shape y in the range

Since all the conditions that make a relation a function are met when transforming shapes, geometric transformations can be considered as functions of shapes.

In the transformation function $y = T(x)$, x is the original shape, T is the transformation applied to the original shape, and y is the image of the shape after transformation.

Students then define their own transformation function, and transform the original shape (the arrow) according to their transformation function. They then do the same thing with shapes of their own. Students must understand that any transformation applied to a shape generates one single possible output (the image).

Exploration 6

An isometric transformation is a transformation that does not modify the shape nor the size of the original shape.

If two points are transformed by the same isometric transformation, then the distance between their images is equal to the distance between the two initial points.

Exploration 7

1 The two single transformations that make up a glide reflection are a reflection in the x-axis followed by a horizontal translation of 8 units to the right. The order of these two transformations can be inversed and the same result will be obtained.

2 The two single transformations (a rotation and a translation) generate a different image when done in a different order. The position of the rotated shape depends on its position before the rotation. If the translation is done before the rotation, its starting position (before rotation) is different than if the translation is done after the rotation. Hence, in this case, the order of the transformations matters.

3 In some combined transformations, the order is not important, and performing one after the other generates the same image regardless of which one is done first. In other combined transformations, the order matters, as the image is different depending on the order the shapes are transformed in.

UNIT 6: A whole range of things

6.1 Univariate statistics

Reflect and discuss 1

- **Is a stem-and-leaf diagram similar to a bar chart?**
  A stem-and-leaf diagram is similar to a bar chart in that it allows you to see the overall distribution of the data. It groups the data visually without losing each individual value, as a histogram or grouped data table would.

- **What are the advantages of keeping raw data?**
  If you want to perform further analysis (calculate other statistics) of the data it is advantageous to keep the raw data.

Reflect and discuss 2

- **What problems might arise when using stem-and-leaf diagrams for large data sets?**
  If the data sets are large it might be better to collect the data into grouped data (bins) as working with a lot of data points can be time consuming and is likely to lead to errors.

- **What about data sets with a small range?**
  If there is only a small range of data, the stem-and-leaf diagram would not show an accurate distribution of the data.

Reflect and discuss 3

- **What does the shape of a back-to-back stem-and-leaf diagram show you?**
  This question starts to answer the conceptual question – how do different representations help you to compare data sets. If you have a back-to-back stem-and-leaf diagram you can 'at a glance' compare the range and mode of the two distributions.

Exploration 1

This exploration is helping the students enquire into the differences between the number of data points and whether the quartile values are actual data points.

1

n	**1a.** $n = 13$	**1b** $n = 14$	**1c** $n = 15$
Q_1	5.5	6	6
Median (Q_2)	10	10.5	11.5
Q_3	17	19	

2

n	**multiple of 4**	**(multiple of 4) +1**	**(multiple of 4) +2**	**(multiple of 4) +3**
Median (Q_2)	not a value in the original data set	yes, a value in the original data set	not necessarily a value in the original data set	yes, a value in the original data set
Q_1 and Q_3	not a value in the original data set	not necessarily a value in the original data set	yes, a value in the original data set	yes, a value in the original data set

3 If the values are the same, the quartile value will be this number.

Reflect and discuss 4

- If n is one less than a multiple of 4 then it is the same as 3 more than a multiple of 4, so Q_1, Q_2 (median) and Q_3 will all be values in the original data set.
- 50% less than or equal to the median
  25% less than or equal to the lower quartile
  75% less than or equal to the upper quartile
- They are called quartiles because the separate the whole data set into quarters (25% in each section).
- Half of the data lies between the first and third data, this is 50%.
- The IQR is a measure of dispersion and it tells you the spread of the distribution. The interquartile range shows the range in values of the central 50% of the data. It can also be used to identify outliers.

Exploration 2

1 10, 22, 30, 32, 38

2

minimum	10
Q_1	22
median	30
Q_3	32
maximum	38

3 In statistics, the range shows how spread a set of data is. The bigger the range, the more spread out the data. If the range is small, the data is closer together or more consistent.

The range of a set of numbers is the largest value (maximum), subtract the smallest value (minimum).

Range = maximum (largest value) − minimum (smallest number)

4 The interquartile range shows the range in values of the central 50% of the data.

To find the interquartile range, subtract the value of the lower quartile, Q_1 (25%) from the value of the upper quartile, Q_3 (75%).

5 On a box and whisker diagram, the whiskers represent the data in the first and fourth quartiles. The box represents the central half of the data.

6 Many answers are possible, for example, 'Number of goals scored in a soccer season by teams in a league'.

Exploration 3

1

	Set 1	**Set 2**
mean	0.54	0.54
mode	0.58	0.58
median	0.58	0.58
IQR	0.1	0.38

2 The values for the mean, median and mode are the same for both sets of data however the IQR is very different. It is possible that set 1 are the people who are fully rested as there is little variation in data, whereas the participants in set 2 have a greater range and IQR and this could be due to the effect of the lack of sleep.

Reflect and discuss 5

- 0.58 is both the mode and the median so is a good representation for the measure of central tendency.
- The IQR and the range would also be important measures.
- The measures of central tendency and the measures of dispersion are both equally important for analysing data.

Exploration 4

1. a

minimum	17
Q_1	21
median	24
Q_3	26
maximum	46

IQR = 26 − 21=5

b 1.5 × 5 =7.5

c $Q_1 - (1.5 \times IQR)$

$Q_1 - (1.5 \times 5)$

21 – 7.5 = 13.5

No values are less than this

$Q_3 + (1.5 \times IQR)$

$Q_1 + (1.5 \times 5)$

26 + 7.5 = 33.5

The value 46 is greater than this value, hence there is one outlier.

2. mean = 24.6

median = 24

mode = 24

	a Including outliers	**b Excluding outliers**
mean	24.6 (3 s.f)	23.4
median	24	24
mode	24	24

range	29	13
interquartile range	5	5

The mean and range are the most greatly affected by the inclusion of outliers.

Reflect and discuss 6

- An outlier is a data point that differs significantly from other observations.
- An outlier may be due to variability in the measurement or it may indicate experimental error. In this case it could have been an error with the measuring equipment, or human error in reading the instrument. If the outlier came from human error it should be excluded from the data set.
- If the outliers are included, it is better to use the median or mode as a measure of central tendency and the IQR as the measure of dispersion.
- The mean and range can lead to inaccuracies and misleading statistical analyses.

UNIT 6: A whole range of things

6.2 Quantifying data

Reflect and discuss 1

- The cumulative frequency value represents the number of times that anything up to and including that value (or group of values) has appeared.
- You need to calculate the cumulative frequency in order to plot a cumulative frequency graph.
- Because the data is grouped, you can find the class interval where the median lies
- In order to estimate the range, you should take the minimum (lower bound) of the smallest group and subtract this from the maximum (upper bound) of the largest group.

Exploration 1

1a

x	f	fx
35	3	105
40	4	160
45	6	270
50	7	350
55	8	440
60	7	630
65	6	390
70	4	280
75	3	225
80	2	160

$$\bar{x} = \frac{\sum fx}{n} = \frac{2800}{50} = 56$$

b

x	f	fx
30	3	90
35	4	140
40	6	240
45	7	315
50	8	400
55	7	385
60	6	360
65	4	260
70	3	210
75	2	150

$$\bar{\bar{x}} = \frac{\sum fx}{n} = \frac{2550}{50} = 51$$

c The mean in part **a** is a whole class interval lower than the mean in part **b**.

d It is better to choose the mid-point of the class interval (the mid-interval value).

2

x	f	**Mid-interval value**	**mid-interval value** $\times f$
$30 < x \leq 35$	3	32.5	97.5
35 etc...	4	37.5	150
	6	42.5	255
	7	47.5	332.5
	8	52.5	420
	7	57.5	402.5
	6	62.5	375
	4	67.5	270

	3	72.5	217.5
	2	77.5	155

$$\bar{\bar{x}} = \frac{\sum fx}{n} = \frac{2675}{50} = 53.5$$

Reflect and discuss 2

- The mid-interval value is the midpoint of the class interval.
  The product of the mid-interval value and the frequency represents the total frequency for that group.
- It is an estimate because you do not know exactly where the data points lie if you have the grouped data. The mid-interval value gives an approximation and estimates the mean.
- If you took the raw data from exploration 1 and calculated the mean, you would get the value: $= 53.652$,which is definitely closer to the estimate using the mid-interval values
- The range of the grouped data would be: $80 - 30 = 50$,
  hereas the range of the raw data would be: $76.5 - 33.5 = 43$
  The range of grouped data will generally be larger than the range of the raw data

Reflect and discuss 3

- The percentiles 25%, 50% and 75% equally divide the dataset into 4 quarters.
- The five-point summary from the raw data is:

minimum	33.5
Q_1	44.2
median	52.35
Q_3	64.1
maximum	76.8

- The five-point summary from the cumulative frequency curve is:

minimum	30
Q_1	44.5
median	53
Q_3	62
maximum	80

- The values are very similar, especially the lower quartile and median.
- As 62 is the value for the upper quartile, you know 75% of the values are less than this. Therefore, you can predict 75% of 50 leaves will be less than this, hence about 38 leaves.
- 25% of the leaves will be greater than this value, hence about 12 leaves.
- As 53 is the median and 25% of the leaves lie between these values, again about 12 leaves.

Exploration 2

1 Each set of results will vary.
2 If the class intervals are larger the results may not be as exact.
3 The smaller the class intervals the closer the measures of central tendencies will be to the raw data, however in this distribution the differences will not be great.
4 The data is uniformly distributed so the distribution will be even
5 Each student will have different graphs and answers.

Reflect and discuss 4

- The choice of class size will affect the measures of central tendency.
- The choice of class size will have less impact on the cumulative frequency curve and five-point summary.
- If there are few class intervals, the data will not be well represented. If there are too many class intervals, the purpose of using grouped data is defeated.

Exploration 3

		mid-interval value	
0 < x < 10	0	5.5	0
11–20	242	15.5	
21–30	19036	25.5	
31–40	48493	35.5	
41–50	12312	45.5	
51–60	1080	55.5	
60	174	65.5	

mean = 34.9
min = 11
lower quartile 35.5
median = 35.5
upper quartile = 35.5
max 70

Reflect and discuss 5

- The best representation is the equal class intervals representation.
- It is best to have between 10 and 15 classes and ensure the class intervals are equal.

UNIT 6: A whole range of things

6.3 Histograms

Reflect and discuss 1

Students should become familiar with characteristics of each type of graph

Favourite colours: bar graph – qualitative data
Pupil to teacher ratio: bar chart – quantitative discrete data
Mid-term examination scores: histogram – quantitative discrete data
Lengths of leaves: histogram – quantitative continuous data

Reflect and discuss 2

Students should become familiar with the differences between rounding with contextual data.

Reflect and discuss 3

Students should be able to determine the shape of the distribution by looking at the class intervals. If there are too many or too few classes this will lead to misrepresentation

Exploration 1

The students explore relative frequency (the definition is given below the exploration).

Students will begin to understand that relative frequency histograms are often a better way for displaying data.

UNIT 7: How do they measure up?

7.1 The right triangle

Reflect and discuss 1

Students should observe that values of $\sin x$ and $\cos x$ lie between 0 and 1, whereas values of $\tan x$ can become very large (since if $0 \leq x < 90, \tan x \geq 0$).

Justification should focus on the idea that the hypotenuse of a right-angled triangle must be longer than both the legs (the perpendicular sides) and hence the ratios $\frac{O}{H}$ and $\frac{A}{H}$ will be between 0 and 1. Since the opposite could be much longer than the adjacent (and vice versa), $\tan x$ could take any positive value.

Exploration 1

The purpose of this exploration is to encourage students to practise good diagram creation, to consider how right-angled triangle trigonometry might be applied in real life, and to see the relationship between similar triangles and the trigonometric ratios. You may also consider the way in which a model may need to simplify real life – the questions rely on measuring the height of the building at a point directly above ground level.

In method 1, students will need to find the distance from the building and the angle of elevation of the top of the building. Potentially they might do this by finding out the angle of elevation of the sun at a certain time of day from data tables or an internet search. Alternatively, they would need to measure the sun's angle of elevation, which requires specialist equipment. Students factoring in their own height may find it easier to simply measure the angle from ground level.
In method 2, students can use similar triangles directly to avoid measuring any angles, though they may find the diagram harder to create (if they do not appreciate that the mirror will lie horizontally on the ground). Of course, in calculating the ratio $\frac{\text{height}}{\text{distance}}$ for the triangle between their position and the mirror, they have found the tangent of the angle of elevation of the building from the mirror.

Reflect and discuss 2

Depending on the way in which students believe they might take the necessary measurements, they might sensibly argue in either direction. The second method requires more measurements, and therefore there is greater potential for error, but the first involves measuring the angle of elevation of the sun, which is inherently tricky.

Exploration 2

Students should use their existing knowledge of the sine, cosine and tangent ratios to find exact expressions for the trigonometric ratios listed in **1c** and **2d**. Question **4** provides an opportunity to reflect on the fact that the largest side in a triangle is opposite the largest angle and that the smallest side is opposite the smallest angle.

Refer students to the 'results worth knowing' table on page 376.

Reflect and discuss 3

Students may reflect that trigonometric ratios are all obtained by dividing one length by another, they may be viewed as the scale factor from one length to another and, as ratios, are necessarily dimensionless (and so have no unit).

Angles of elevation and depression are equal because they are both measured from the horizontal and hence are alternate angles.

Reflect and discuss 4

Students should observe that, as one approaches an object, its angle of elevation increases. This will help them construct diagrams in cases where, for example, a building is observed from two points, one closer than the other. Correspondingly, as the sun rises in the sky, the shadow shortens as the angle of elevation increases.

Reflect and discuss 5

Since problems involving trigonometry involve inferring some measurements from others, we can find distances over obstacles, or that would be otherwise hard to physically measure. Problems involving non right-angled triangles can be decomposed into simpler shapes (right-angled triangles). In due course, students may also encounter the sine rule and cosine rule.

Bearings are given according to a convention to ensure that there is no ambiguity when giving instructions, which could be catastrophic if dealing with the navigation of ships or planes, for example.

Exploration 3/Reflect and discuss 6

Students should revisit the ideas discussed in Reflect and discuss 1.

To justify why $\tan 90$ is undefined (and hence why the calculator gives an error), students should try to construct a triangle with two right angles. The 'opposite' side and the 'hypotenuse' will never meet, and so the 'opposite' will be infinitely long.

Exploration 4

If it has not already been drawn out in previous conversation, this exploration allows students to observe that $\sin\theta \to 0$ and $\cos\theta \to 1$ as $\theta \to 0$ and $\sin\theta \to 1$ and $\cos\theta \to 0$ as $\theta \to 90$.

Exploration 5

Depending on the accuracy of their measurements, students should notice that the gradient of a line is the tangent of the angle that it makes with the horizontal. To justify it, they should reflect that gradient is given by $\frac{\text{change in } y}{\text{change in } x}$ between any pair of points on a line and that these two lengths may easily found by constructing a triangle with horizontal and vertical sides – where the angle opposite the vertical side is exactly the angle between the line and the horizontal, as required. Weaker students might be well served to omit the final questions with negative gradient.

UNIT 7: How do they measure up?

7.2 Properties of circles

Reflect and discuss 1

A secant, being a line, is infinitely long, whereas a chord, being a line segment, is entirely contained within the circle.

Exploration 1

Students are expected to first observe, and then justify, the result which appears in the subsequent definition box: that within a circle, the triangle formed by two radii and a chord is isosceles, and, as with all isosceles triangles, bisecting the base generates two right-angled triangles.

Exploration 2

It is not anticipated that students will memorise the formula $\theta = 2\arcsin\left(\frac{c}{2r}\right)$, but rather that they will learn the approach of using right-angled triangle trigonometry to find the central angle subtended by a chord.

Exploration 3

The object of this exploration is to help students see that while the 'number of slices' is sometimes a convenient measure for how a circle has been divided, the proportion of the circle that constitutes an individual sector or arc is also a useful measure, especially when dealing with sectors where the central angle is not a factor of 360. The most important results for them to focus on are the arc length and sector area in the final row. The completed table is shown below; students might discuss how meaningful the idea of 'sector perimeter' is in the case of the undivided circle.

Number of equal sectors	Fraction	Central angle	Arc length	Sector perimeter	Sector area
1	1	360	$2\pi r$	N/A	πr^2
2	$\frac{1}{2}$	180	$\frac{2\pi r}{2}$	$\pi r + 2r$	$\frac{\pi r^2}{2}$

3	$\frac{1}{3}$	120	$\frac{2\pi r}{3}$	$\frac{2\pi r}{3}+2r$	$\frac{\pi r^2}{3}$
4	$\frac{1}{4}$	90	$\frac{2\pi r}{4}$	$\frac{\pi r}{2}+2r$	$\frac{\pi r^2}{4}$
5	$\frac{1}{5}$	72	$\frac{2\pi r}{5}$	$\frac{2\pi r}{5}+2r$	$\frac{\pi r^2}{5}$
6	$\frac{1}{6}$	60	$\frac{2\pi r}{6}$	$\frac{\pi r}{3}+2r$	$\frac{\pi r^2}{6}$
8	$\frac{1}{8}$	45	$\frac{2\pi r}{8}$	$\frac{\pi r}{4}+2r$	$\frac{\pi r^2}{8}$
9	$\frac{1}{9}$	40	$\frac{2\pi r}{9}$	$\frac{2\pi r}{9}+2r$	$\frac{\pi r^2}{9}$
10	$\frac{1}{10}$	36	$\frac{2\pi r}{10}$	$\frac{\pi r}{5}+2r$	$\frac{\pi r^2}{10}$
30	$\frac{1}{30}$	12	$\frac{2\pi r}{30}$	$\frac{\pi r}{15}+2r$	$\frac{\pi r^2}{30}$
36	$\frac{1}{36}$	10	$\frac{2\pi r}{36}$	$\frac{\pi r}{18}+2r$	$\frac{\pi r^2}{36}$
40	$\frac{1}{40}$	9	$\frac{2\pi r}{40}$	$\frac{\pi r}{20}+2r$	$\frac{\pi r^2}{40}$
n	$\frac{1}{n}$	$\frac{360}{n}$	$\frac{2\pi r}{n}$	$\frac{2\pi r}{n}+2r$	$\frac{\pi r^2}{n}$
$\frac{360}{\theta}$	$\frac{\theta}{360}$	θ	$\frac{\theta}{360}\times 2\pi r$	$\frac{\pi r\theta}{180}+2r$	$\frac{\theta}{360}\times \pi r^2$

Reflect and discuss 2

Students might wish to consider circumstances where accuracy is very important, but where practicality is also a factor. For example, if one were to consider using mosaic tiles to cover the inside of a circular swimming pool, what error would result from taking π to be 3.14? And how would that error compare to errors down to variation in the spaces between the tiles?

For comparison, NASA uses values for π correct to 15 or 16 decimal places (https://blogs.scientificamerican.com/observations/how-much-pi-do-you-need/).

Exploration 4

This exploration gives students the opportunity to practice rearranging formulae. Once complete, you may encourage them to reflect on whether it is advantageous to memorize these formulae, or simply to rearrange them when needed. They might also consider, having tackled the following Practice questions, whether it is useful to rearrange the formula prior to use, or to substitute in the values first of all.

1 $\theta = \frac{180l}{\pi r}$

2 $\theta = \frac{360A}{\pi r^2}$

3 $l = \frac{\pi r\,\theta}{180}$

4 $r = \frac{180l}{\pi\theta}$

5 $A = \frac{\pi r^2\theta}{360}$

6 $r = \sqrt{\frac{360A}{\pi\theta}}$

UNIT 7: How do they measure up?

7.3 Circle theorems 1

Many of the explorations in this section benefit from the use of dynamic geometry software; there are many tools freely available online. You may find it easier to form some of the diagrams for your students rather than getting them to form their own, though it is certainly a skill that might help them with other exploration or investigation in the future.

Exploration 1

Students should find that they are able to create tangents by eye and then measure the angle between tangent and radius to be 90°. If they use a tangent creation tool, it will take away any opportunity to vary the angle.

Exploration 2

Students should observe that the angle at P is always 90°. They would not be expected at this stage to use language such as 'subtended by the diameter'.

Exploration 3

Students should observe that varying P (whilst holding the chord AB constant) means that angle $\angle APB$ remains constant. You might prompt them to notice that Exploration 2 therefore shows a special case of this result.

Exploration 4

Students should observe that opposite angles in a cyclic quadrilateral are supplementary.

Exploration 5

Students should observe that angle $\angle AOB$ is twice $\angle APB$. The question prompts them to keep P on the major arc, though you might ask stronger students to investigate what happens as P moves to the minor arc and to reflect on how this relates to the cyclic quadrilaterals in Exploration 4.

Exploration 6

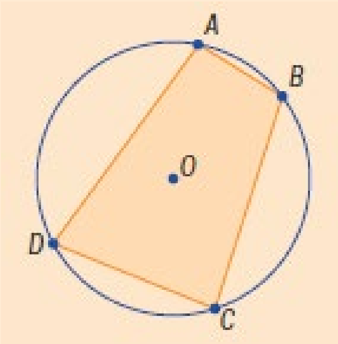
This exploration guides students towards conjecturing the Alternate Segment Theorem; they have two opportunities to detect it within the triangle.

Exploration 7

This exploration gives students the opportunity to complete a proof of the result that opposite angles in a cycle quadrilateral are supplementary. They should complete the skeleton proof; a possible completion is provided here.

Proof

Consider a cyclic quadrilateral *ABCD* inscribed in a circle with center *O*.



OA = *OB* = *OC* = *OD* because ***they are all radii.***

Therefore Δ*OAB*, **Δ*OBC* , Δ*OCD*** and **Δ*ODA*** are all ***isosceles*** triangles.

Therefore ∠*OAB* = ∠*OBA*, **∠*OBC*** = **∠*OCB***, **∠*OCD*** = **∠*ODC***, and **∠*ODA*** = **∠*OAD*** because ***base angles in isosceles triangles are equal.***

$\angle OAB + \angle OBA + \angle \mathbf{OBC} + \angle \mathbf{OCB} + \angle \mathbf{OCD} + \angle \mathbf{ODC} + \angle \mathbf{ODA} + \angle \mathbf{OAD} = 360°$

$\Rightarrow 2(\angle OAB + \angle OAD + \angle \mathbf{OCD} + \angle \mathbf{OCB}) = 360°$

$\Rightarrow 2(\angle DAB + \angle \mathbf{DCB}) = 360°$

$\Rightarrow \angle \mathbf{DAB} + \angle \mathbf{DCB} = 180°$

Therefore the angles at *A* and *C* are supplementary, and the angles at *B* and *D* are supplementary.

There are two other cases which should be considered. The first is the case when two adjacent points on the quadrilateral are diametrically opposed on the circle, i.e. an edge of the quadrilateral passes through the centre of the circle. In this case, one of the three 'isosceles triangles' is a straight line. The second is when the whole cyclic quadrilateral lies inside a semicircle and so the isosceles triangles are wholly or partially exterior to the quadrilateral.

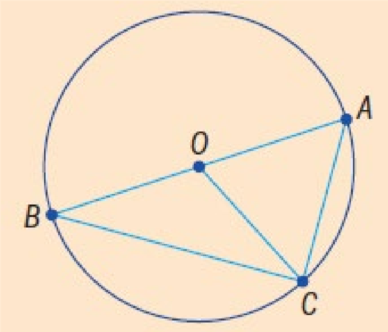
These cases can either be proved as corollaries of the given proof (using, if you have proved it already, the fact that angles subtended by the same chord are equal; a diagonal of the quadrilateral may be held constant and the other points moved to be in opposite semicircles) or by starting with fresh diagrams, which is conceptually simpler but more time-consuming.

It is probably the case (because of time constraints) that only the strongest students will prove all cases and most will have to take the result on trust.

Exploration 8

Again, a sample answer is given.

Proof



Consider a circle with center O where AB is a diameter. Let C be any point on the circumference of the circle.

ΔOAC and ΔOCB are ***isosceles*** triangles.

Hence $\angle OCA = \angle \boldsymbol{OAC}$ and $\angle OCB = \angle \boldsymbol{OBC}$

$\angle ACB = \angle OCA + \angle OCB$

$\angle AOC = \angle \boldsymbol{OBC} + \angle \boldsymbol{OCB}$ and $\angle BOC = \angle \boldsymbol{OAC} + \angle \boldsymbol{OCA}$

$\Rightarrow \angle AOC = 2\angle OCB$ and $\angle BOC = 2\angle OCA$

$\Rightarrow \angle AOC + \angle BOC = 2(\angle \boldsymbol{OCB} + \angle \boldsymbol{OCA})$

$\Rightarrow \angle AOC + \angle BOC = 2\angle \boldsymbol{BCA}$

$\Rightarrow 2\angle \boldsymbol{BCA} = 180^\circ$

$\Rightarrow \angle \boldsymbol{BCA} = 90^\circ$

The option of considering $\angle AOC$ and $\angle BOC$ as exterior angles to the other triangle may be tricky for students to spot; they could construct alternative proofs by considering the sum of a triangle's interior angles.

It is also possible to prove this result by considering a point *D* which is the reflection of *C* in *AB*. Then the angles at *C* and *D* must be equal, and since *ACDB* is a cyclic quadrilateral, the angles at *C* and *D* are also supplementary. Equal supplementary angles must be right angles. You might ask students which proof they prefer.

Exploration 9

If students have understood Exploration 7, and if you happen to have discussed the alternative proof suggested in the commentary to Exploration 8, this should be relatively straightforward. Since both angles are supplementary to ∠PRQ, they must be equal.

Reflect and discuss 1

Of these four statements, only the third statement is false, though students might not notice it. It is worth stressing that the consequent part (the ‘then’ in an ‘if … then …’ statement) must be complete. Certainly, if $x^2 = 4$ then it is possible that $x = 2$, but this is not a complete statement, and so therefore it is not **the** logical consequence of the statement.

Reflect and discuss 2

The first of these is deceptive. At first glance, it may seem that, since any three points form a triangle, this result must be true. However, three points only form a triangle if they are not collinear, which indeed they could be even if the sum of the given angles were 180°. So, in fact, the statement only holds true in the case that the points are not collinear, in which case the angle fact is redundant.

The second is easily demonstrated false by means of a counterexample (indeed π is a fine counterexample since it meets the brief of not being prime, though you might be better served to stick to examples such as $x = 6$). The other two statements are true.

It therefore follows that a statement and its converse can be true or false independently of one another

Exploration 10/Reflect and discuss 3

Students should find that the point being traced creates a semicircle centered on the midpoint of AC. In this sense, the diagram certainly supports the claim that the converse is true. Convincing as the students may find it, however, it does not constitute a proof.

Exploration 11

This exercise is a good time-filler for able students, who will need to give careful consideration both to what the converse actually states and also to how they could create such a diagram using dynamic geometry software. If they are unable to create dynamic versions, then they may at least be able to draw examples with measured angles

7.4 Circle theorems 2

Exploration 1/Reflect and discuss 1

By the end of step 4, students should have observed that if *AXB* and *CXD* are chords in the same circle, then $AX \times BX = CX \times DX$. In step 5, they should find that it holds regardless of the radius, but that (if nothing else) their diagram fails if *AB* and *CD* do not intersect interior to the circle. If they have constructed lines *AB* and *CD* rather than line segments, they may discover the other cases of the theorem immediately! If students are having difficulties with the diagram, it may well be that they have drawn points *ABCD* in order on the circumference and then erroneously joined *A* to *C* and *B* to *D*.

It is hoped that students will appreciate that a set of examples (such as they have generated in their exploration) does not constitute a proof. Stronger students might consider applications of similar triangles to attempt a proof.

Exploration 2

Students may be able to modify their previous diagram, though it is probably just as easy to start from scratch and will give them good practice at using dynamic geometry software. In step 2, the most obvious exception would be when the secants are parallel, and so the intersection does not exist. Students might also wonder whether the theorem makes sense if, for example, *B* and *C* are coincident. If they are, then both *BX* and *CX* would be 0, and hence the theorem would be true, though of little practical use.

Exploration 3/Reflect and discuss 2

In this exploration, students can definitely reuse their diagram from Exploration 2 to save time. As *B* approaches *A*, they should find that the secant starts to look more like a tangent, however, they won't be able to make *A* and *B* coincide without the diagram losing the line *AB*. For that reason they may wish to work algebraically to consider that if $AX \approx BX$ then $CX \times CX \approx AX^2$.

Students may also observe that the secant approaches the tangent and therefore decide to redraw their diagram with a tangent generated by the computer. These steps are prompted by the points in Reflect and discuss 2, and summarized beneath the exploration.

UNIT 8: What comes next?

8.1 Sequences

Exploration 1/Reflect and discuss 1

1 Students may well come up with a wide range of descriptions; since a sequence does not necessarily follow a pattern, any possible continuation is valid – for example, though the first sequence looks to be an arithmetic sequence, there is no reason why it could not be periodic (1, 3, 5, 7, 9, 1, 3, 5, 7, 9, 1, 3, 5, 7, 9, …) or, indeed, be a list of randomly generated integers. The sort of answers student might give are:

- **a** They are the odd numbers; they go up by two every time.
- **b** They are the square numbers, they go up by 3, 5, 7, 9 etc.
- **c** They go up by three every time; they are four more than the multiples of three.
- **d** The sequence alternates between 1 and 2; it repeats; it is periodic.
- **e** It doubles every time; the numbers are three times the powers of 2.
- **f** The numbers seem to be random; there is no pattern. It is impossible to predict/any prediction would be valid.

2 The given digits are the result of taking the nth digit of π. The sequence 3, 1, 4, 1, 5, could therefore easily continue by alternately growing and returning to 1, as in the first line, though it is just as valid for the sequence to continue to take the nth digit of π, as in the second line.

In the Reflect and discuss prompts, it may start to become clear to British readers that the sequence appears to be a phone number (it is Oxford University Press's number). You may wish to substitute your own school's telephone number for the digit in **1f** if you prefer – though experience suggests that even when presented in this way students still fail to recognise phone numbers in this context because they are so unexpected for them. The digits certainly form a sequence, though it would probably be a stretch to use the word 'pattern'. It is worth noting that it would be basically impossible for anyone to recognize the sequence here. The first five numbers are simply a dialling code, so it could well be any phone number in Oxford. Only, in this case, by knowing the whole sequence is it possible to determine its significance, and hence if one digit were omitted, it could not with confidence be restored.

Reflect and discuss 2

Using *a, b, c, d, e* to label the terms of a sequence would be problematic for a number of reasons: it is difficult to identify which number term in the sequence would relate to which letter; you will run out of letters quickly, and then need to use other cases or alphabets; you may wish to use the letters for other purposes in a question, or refer to more than one sequence (e.g. a_i and b_i).

Exploration 2/Reflect and discuss 3

In step 1, students may spot that they can do this either by using the formula every time, or by adding a fixed amount to the previous term.

a 10, 13, 16, 19, 22
b 1, −1, −3, −5, −7
c 3, 8, 13, 18, 23
d 9, 13, 17, 21, 25

In steps 2 and 3, students should notice that the first differences are constant, that they match the coefficient of n in the formula and, vitally, that each of the four given formulae are linear. Were they not linear, the differences would vary – a fact easily demonstrated by using, for example, $e_n = n^2$.

The graphs generated would consist of a set of points lying on a straight line. Students will be likely to plot any of the relationships of the form $u_n = a + bn$ on their GDCs or graphing software as $y = a + bx$. Doing so will create a line and it is a good opportunity to discuss the difference between discrete and continuous information - you might start by asking students what they think the value of y represents when $x = 2.5$.

Exploration 3/Reflect and discuss 4

The objective of this exploration is for students to see that there is a link between linear sequences and linear functions, with which they should already have a degree of familiarity. Reducing the size of the charging period (to 15 minutes, 10 minutes, or less) will mean that the staircase graph (the right-hand of the two provided) will be made of smaller steps. As the steps continue to get smaller, the graph will become smaller until, eventually, it will appear smooth. Of course, even if the hire company were simply to charge at a rate (rather than charging per interval), the graph would not be smooth in reality, because time (especially for boat hire) can only be measured to a reasonable degree of accuracy, and the amounts charged will need to be payable in dollars and cents – so the problem never truly becomes continuous.

UNIT 8: What comes next?

8.2 Rearranging formulae and proportion

Exploration 1/Reflect and discuss 1

Students are already likely to be familiar with rearranging equations. It is important that they appreciate the requirement not only that a variable appears on its own on one side of the equation, but also that it does not appear anywhere else.

Exploration 2

This exploration is an introduction to the idea of direct proportion in the context of a scenario in which students should intuitively understand the model. The growth in cost when purchasing an increasing number of items at a fixed cost per unit is both highly calculable and highly comprehensible.

Students should associate fixed-rate problems with straight-line graphs passing through the origin.

Reflect and discuss 2

The graph passes through the origin because no bulbs cost no money. If, for example, there was a discount for buying in bulk then the situation would no longer be one where direct proportion would be observed.

Students may comment on the discrete nature of the scenario and the problems inherent there with graphing the solution; it is a fair question. You could sensibly ask what sorts of situations might avoid this problem.

Reflect and discuss 3

Students that have understood the idea of a proportionality constant, should easily identify that the cost per bulb is the appropriate value in each case. The idea of a constant ratio between x and y is easily demonstrated by making k the subject of the proportionality equation.

Reflect and discuss 4

The proportionality constant is the gradient. Models of direct proportion are of the form $y = kx$; if x is 0 and y is not, then we have a multiple of 0 which is non-zero – a contradiction.

Exploration 3/Reflect and discuss 5

This is a nice variation on the traditional ‘It takes 3 builders 4 days to build a wall’ problem. By considering both the time required to complete a fixed job, and the amount paid per hour to the entire workforce, pupils generate both a proportional relationship and also an inversely proportional relationship in the same context. It is useful to encourage students to consider a variety of situations in which inverse proportion arises naturally. Typically, situations where a fixed-size task can be completed at a varying rate, or by varying numbers of people.

In the following discussion, if students have appreciated the role of the ‘fixed-size task’, they should be able to see that the scale of this task is what determines the constant of proportionality

Exploration 4

Students already familiar with scale factors may know that (area scale factor) = (length scale factor)^{2}and corresponding results for volume. They may recognize this in the relationship between the radius and area of a circle, i.e. that doubling the radius increases the area by a factor of 4. It is hoped that they will see that this doesn’t meet the requirements of direct proportion between *A* and *r*, but that there is direct proportion between *A* and r^2.

Exploration 5/Reflect and discuss 6

In this task, students can start by ignoring the practicalities of tiling and instead imagine that tiles can be used completely. In step 2, they should note that since $n = \frac{A_w}{A_t}$, it follows that $n = \frac{A_w}{l^2}$, and hence *n* is inversely proportional to the square of l. With smaller tiles, $m = \frac{9A_w}{4l^2}$, and so we note that $m \propto n$. This is because 9 new tiles are equivalent to 4 old tiles, i.e. the relationship between the number needed is linear.

Students will hopefully see that this process gives an underestimate for the number of tiles needed to solve a practical problem. Not only will there be overspill, and partial tiles running off the end of the area to tile, any student with a bit of DIY experience will know that there will inevitably be some tiles broken accidentally as Diego covers the wall!

A nice observation is that the model becomes more accurate as tile size becomes smaller (since the quantity wasted will diminish).

Exploration 6

The goal here is to help students consider an alternative way of dealing with proportionality problems. Consider that if $y \propto x$, then y doubles as x doubles. In real life this is a far easier way of working with simple proportionality situations than forming an equation:
Work twice as long? … Get paid twice as much. Three times as far to travel? … It will take three times as long.

Equally, most people implicitly deal with simple inverse proportion problems in real life by inverting the scale factor:
Twice as many workers? … Halve the time needed. Travelling three times as fast? … It will only take a third of the time.

Such intuition tends to be weaker for non-linear relationships, but it is just as valid:
Double the radius of the circular lawn? … Quadruple the time taken to mow it. Double the diameter of the bottle? … Octuple its capacity.

Inverse power relationships are less prevalent, but students may be aware of some inverse square or cube laws from physics.

Reflect and discuss 7

Students may well offer a wide range of (correct) answers here and there is certainly no single way of answering the question. A good answer based on Exploration 6 would be to observe that situations where, if one variable grows by a scale factor c, the other grows by c^n, where n is a positive constant, a direct variation relationship exists. Similarly, if the second quantity shrinks by a factor of c^n, inverse proportion occurs.

Generally, if people say 'proportional' without any qualifying information, they mean directly proportional.

Exploration 7/Reflect and discuss 8

The objective here is to build familiarity with the important graph shapes associated with proportionality in all its forms. Students should observe that direct relationships always pass through the origin, growing more quickly as either k or n is increased. Inverse relationships will always have an asymptote at $x = 0$, and will become taller (or appear to move further from the origin) as k increases, approaching their horizontal asymptote faster for larger n.

Exploration 8

In the situations described here, the *cost of coffee* is always proportional to the quantity consumed, but the *total cost* to the user is not, because of fixed costs in the bring-your-own mug version. Students might usefully consider the point at which one option becomes more cost-effective than the other. Any situation that has fixed start-up costs would satisfy the requirements of step 4.

Exploration 9/Reflect and discuss 9

Depending on what material students have previously covered, they may interpret this very quickly as a matter of reflecting graphs in the x-axis. If not, it is a good moment to mention the idea of graph transformations quickly.

UNIT 9: So, what do you think?

9.1 Sampling techniques

Reflect and discuss 1

The purpose of this exercise is to introduce the students to the concept of generalization. Students should look at the measures of central tendency and measures of dispersion.

Some generalizations may include:

- The mode of Crescent Moon is lower than the mode of Oyster Bay
- Crescent Moon is unimodal whereas Oyster Bay is bimodal
- The range of Crescent Moon is much larger than Oyster Bay

Reflect and discuss 2

The median of Crescent Moon is 73 whereas the median of Oyster Bay is 92

The generalizations stated could be argued in either direction. It is important for students to realize that generalizations can have different pieces of evidence to support them.

Reflect and discuss 3

It is important to gather evidence and be very clear, coherent and concise when you make a generalization.

Exploration 1

1 Sampling the whole population of the company can be very time-consuming and not necessary to get the required information.

2 The students should look for the advantages and disadvantages of these methods.

Reflect and discuss 4

The students should be able to make generalizations about the sampling techniques and can research online benefits and limitations.

Reflect and discuss 5

It is important that students become familiar with the notion of bias and how it can affect the results of a survey. They should start to understand the difference between random and non-random sampling.

Exploration 2

1 $\bar{\bar{x}} = 182.125\ \text{cm}$
2 $\bar{\bar{x}} = 176.125\ \text{cm}$
3 The statistic appears to support the claim, though it is important to understand whether the sample of 16 boys is representative of the year group.

Reflect and discuss 6

Students have generalized by using the mean of the sample of 16 to represent the mean of the year group. Whether this seems reasonable or not might depend on the size of the year group.

Exploration 3

Students should take time to research data and find a way of sampling. They can then investigate patterns and trends for the data.

Exploration 4

The students should see how they can make different statistical representations to make different generalizations.

Reflect and discuss 7

If the employees are consistent, then in this case, they would produce the same amount of components each day

The range and interquartile range would be good indicators for consistency.

Exploration 5

The aim of this exploration is for students to appreciate that a larger sample will better reflect the population, and that samples (being random) will not necessarily have the same summary statistics

.

Reflect and discuss 8

Smaller sample sizes may be used if the sample will exhaust the product or damage the environment.

9.2 Bivariate data

Exploration 1

The purpose of this exploration is to begin to look at relationships within data sets. As the mathematics score increases the physics score is also higher which might suggest there is a relationship between the two variables.

It is always important to note that the students should see that there are exceptions and not each student will follow this relationship.
In this case when they draw a line of best fit it will have a positive gradient and the points are all close to line.

Exploration 2

If the line passes through the points (2, 2) and (5, 7), then the line will have a gradient of $\frac{5}{3}$ and the equation $y = \frac{5}{3}x - \frac{2}{3}$

In the second question, the line of best fit on the graph has a *y*-intercept at 5 and the equation of the line would be

$$y = -\frac{25}{53}x + 5$$

This means that if a tadpole is 7.9 weeks old, then the length of its tail would be approximately 1.27 mm

Reflect and discuss 1

Even though the points are close to the line, indicating that it is an accurate line of best fit, if a tadpole was 15 weeks old, the value would be negative. In the context of the question this would not make sense. Perhaps the tadpole is now a frog!